

# Geometry of quadratic differential systems in the neighbourhood of infinity

Dana SCHLOMIUK\*

Département de Mathématiques et de Statistiques  
Université de Montréal

Nicolae VULPE†

Institute of Mathematics and Computer Science  
Academy of Science of Moldova

## Abstract

In this article we consider the behavior in the vicinity of infinity of the class of all planar quadratic differential systems. This family depends on twelve parameters but due to action of the affine group and re-scaling of time the family actually depends on five parameters. We give simple, integer-valued geometric invariants for this group action which classify this family according to the topology of their phase portraits in the vicinity of infinity. For each one of the classes obtained we give necessary and sufficient conditions in terms of algebraic invariants and comitants so as to be able to easily retrieve for any system, in any chart, the geometric as well as the dynamic characteristics of the systems in the neighborhood of infinity. A program was implemented for computer calculations.

**Keywords:** Poincaré compactification, singular point, phase portrait, topological index, intersection multiplicity, linear group, affine invariant.

---

\*Work supported by NSERC and by the Quebec Education Ministry

†Partially supported by NSERC

# 1 Introduction

We consider real planar polynomial differential system, i.e. systems of the form

$$\frac{dx}{dt} = p(x, y), \quad \frac{dy}{dt} = q(x, y) \quad (S)$$

where  $p$  and  $q$  are polynomials in  $x$  and  $y$  with real coefficients ( $p, q \in \mathbb{R}[x, y]$ ). In this article, a system of the above form with  $\max(\deg(p), \deg(q)) = 2$  will be called quadratic.

These are the simplest nonlinear differential systems. However, global problems regarding this class are difficult to solve. In 1900 Hilbert gave his list of 23 problems and one of the few of them still unsolved, the second part of Hilbert's 16th problem, is on planar polynomial systems. This problem which asks for the maximum  $H(n)$  of the numbers of limit cycles occurring in differential systems with  $\max(\deg(p), \deg(q)) \leq n$  (for a discussion of this problem cf. [25]), is still unsolved even for quadratic differential systems. The interest is in the global behavior of all solutions in the whole plane and even at infinity (cf. [10]) and this for a whole family of systems, which is why this problem is so hard. The set **QS** of quadratic differential systems depends on 12 parameters, the coefficients of the two polynomials  $p$  and  $q$ . On **QS** acts the group of affine transformations and of changes of scale on the time axis. So the space actually depends on five parameters. But even five is a large number considering that we expect this class to yield thousands of distinct phase portraits. For this reason people have attempted to study particular classes of quadratic systems and for some classes a complete classification of phase portraits with respect to topological equivalence was obtained (quadratic systems with a center [31], [23], [18], quadratic Hamiltonian systems [1], [7], quadratic chordal systems [9], quadratic systems with a weak focus of third order [2] and [14], etc.).

The goal in most of these articles was to obtain all topologically distinct phase portraits for the particular class considered. Two systems  $(S)$  and  $(S')$  are topologically equivalent if there exists a homeomorphism  $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$  such that  $f$  carries orbits to orbits preserving (or reversing) their orientation. In most articles, the classifications were done by using specific charts and normal forms for the systems in these charts with respect to parameters satisfying certain inequalities or equations. The results are not readily applicable for systems given in normal forms with respect to other charts. This dependence on a specific normal form yields results which are not geometric. Indeed, ever since Klein gave his famous Erlangen program, we are used to calling a property geometric, if it is invariant under the action of some group. In this sense, most of the results obtained are not geometrical since they are not independent of the charts considered.

The orbit space of **QS** under the group action is five-dimensional. The global study of the class **QS** makes it necessary to use normal forms in several charts. To obtain the global picture, we need to be able to glue these charts and for this gluing process invariants are helpful. Furthermore in work in progress on this space, invariants are very helpful even in choosing the charts.

Chart independent classification results were obtained by K.S. Sibirsky and his school (cf. [30], [22], [7]) using the algebraic invariant theory of differential equations developed by Sibirsky and his disciples (cf. [28], [29], [32], [21]). However most of the articles of the school of Sibirsky were published in Russian, only some appeared in translations which partly explains why this theory is rather unknown in the west. In these articles, invariants and comitants are introduced in their multi-index tensorial form, certain rather artificial polynomial combinations of these are chosen and classifications are given in terms of these combinations. In the end these classifications remain insufficiently related to the geometry of the systems.

We need much simpler invariants, simpler than the configuration space of Markus (cf.[15]), possibly even integer valued invariants which could convey to us in simple terms properties of the global geometry of the systems. We would also need a way of computing at least some of these simple invariants for any system in whatever chart it may be given to us.

In [18], [14] the authors gave topological classifications in terms of the global geometry of the classes of systems considered. These classifications are affinely invariant and they are expressed in [18] in terms of the geometry of algebraic invariant curves of the systems considered or in terms of very simple integer-valued invariants in [14] and [26] reflecting the geometry of the systems. There is however a need to have an efficient way of effectively computing, independent of charts these simpler integer-valued , geometric invariants.

In spite of their awesome character, polynomial invariants and comitants are a very powerful computational tool applicable to any canonical form and they can be programmed on a computer. There is thus a need to merge the geometric methods above mentioned with the algebraic invariant approach. In this work we propose to do just that for the specific problem of classifying topologically quadratic systems in the neighbourhood of infinity.

In [13] Kooij and Reyn obtained all possible local phase portraits around a single singular point at infinity of an arbitrary quadratic vector field. In [13] they did not consider the possible ways of combining such singularities so as to obtain a topological classification of

quadratic systems in a neighborhood of the line at infinity. In [16] I. Nicolaev and N. Vulpe obtained such a classification in terms of algebraic invariants and comitants and in [4] the affine invariant classification of quadratic system with respect to the possible distributions of the multiplicities of singularities at infinity was obtained by V. Baltag and N. Vulpe [4]. These classifications use the technical language of algebraic invariant theory developed by the school of Sibirsky ([28],[32],[6], etc).

The goal of this work is to combine the geometric approach in [18], [14] and [26] with the algebraic invariant approach in [16] and [4] for the topological classification of quadratic systems in the neighborhood of infinity. We need simplicity and clarity in the geometric classification as well as applicability to any particularly chosen chart. In this article, which is based on [27] we introduce the notions and prove the necessary results which permit this in as self-contained a way as possible. We also point out that in the attempt to merge the geometric invariants with the algebraic ones, the geometry led us to simpler algebraic invariants than those in [16] yielding simpler conditions in the classification Theorem 7.1.

This work could be applied along with an analogous one for finite singularities, as an initial step, to the problem of classifying all quadratic differential systems.

The article is organized as follows: In §2 we consider the two compactifications of real planar polynomial systems and the foliations with singularities, real and complex, on the real and complex projective planes, associated to these systems.

In §3 we describe the purely geometric objects, i.e. the divisors attached to the line at infinity, introduced in [26], which encode the multiplicities at infinity of the systems, and attach to these some integer-valued global affine invariants.

In §4 we consider group actions on quadratic differential systems and define algebraic invariants and comitants with respect to these group actions. We also give, using a comitant, canonical forms for these differential systems according to their behavior at infinity.

In §5 we state and prove the classification theorem (Theorem 5.1) of the quadratic differential systems according to their multiplicity divisors at infinity and for each class we give the necessary and sufficient conditions in terms of algebraic invariants and comitants with respect to the group action. These conditions allow us to compute for any system and in any chart the types of the multiplicity divisors associated to the system.

In §6 we introduce new classifying tools, among them the index divisor encoding globally the topological indices of the singularities at infinity of any polynomial differential system without a line of singularities at infinity. We also introduce a divisor encoding globally the

number of local separatrices bounding a hyperbolic sector of a singular point at infinity.

In §7 we state and prove the topological classification theorem (Theorem 7.1). This classification is expressed on one side in terms of geometrical, affine integer-valued invariants, which convey in simple terms the geometric and dynamic properties of the systems according to their behavior in the vicinity of infinity; on the other hand in terms of algebraic invariants and comitants. In the end we are able to read for any system and in any chart, its geometric and dynamical properties at infinity once these algebraic invariants and comitants are calculated. These calculations could be done on a computer. A complete dictionary of integer-valued geometric invariants and polynomial invariants is given.

In the Appendix we list the invariants and comitants used in [16] and which are needed for the proofs of the main results. These are also listed for the purpose of comparison with the simpler algebraic invariants and comitants used in this article. Highlighting the geometry of the systems via the integer-valued invariants introduced, helped us to choose better algebraic invariants and comitants than those in [16], closer to the geometry of the systems.

## 2 The two compactifications of real planar polynomial vector fields

A real planar polynomial system  $(S)$  can be compactified on the sphere as follows: Consider the  $x, y$  plane as being the plane  $Z = 1$  in the space  $\mathbb{R}^3$  with coordinates  $X, Y, Z$ . The central projection of the vector field  $p\partial/\partial x + q\partial/\partial y$  on the sphere of radius one yields a diffeomorphic vector field on the upper hemisphere and also another vector field on the lower hemisphere. Poincaré indicated briefly in [20] that one can construct an analytic vector field  $\mathcal{V}$  on the whole sphere such that its restriction on the upper hemisphere has the same phase curves as the one induced by the phase curves of  $(S)$  via the central projection. A complete proof was given much later in [10]. The analytic vector field  $\mathcal{V}$  on the whole sphere obtained in this way is called the Poincaré field associated to the system  $(S)$ . The phase curves of  $\mathcal{V}$  coincide in each chart with phase curves induced by planar polynomial vector fields, in particular in the chart corresponding to  $Z = 1$ , denoting the two coordinate axes  $x, y$  corresponding to the  $OX$  and  $OY$  directions, they coincide with the phase curves induced by  $(S)$ . The two planar polynomial vector fields  $U, V$  associated to the charts for  $X = 1$  (with local coordinates  $(u, z)$ ) and for  $Y = 1$  (with local coordinates  $(v, w)$ ) and

changes of coordinates  $u = y/x$ ,  $z = 1/x$ , or  $v = x/y$ ,  $w = 1/y$  are as follows:

$$U \begin{cases} \frac{du}{dt} = C(1, u, z), \\ \frac{dz}{dt} = zP(1, u, z), \end{cases} \quad \text{and} \quad V \begin{cases} \frac{dv}{dt} = C(v, 1, w), \\ \frac{dw}{dt} = -wQ(v, 1, w), \end{cases}$$

where  $P, Q$  and  $C$  are defined further below.

By the compactification of the planar polynomial vector field associated to  $(S)$  we understand the restriction  $\mathcal{V}|_{\mathcal{H}'}$  (where by  $\mathcal{H}'$  we understand the upper hemisphere  $\mathcal{H}$  completed with the equator) of the analytic vector field  $\mathcal{V}$  on the sphere. In this work we are interested in the topological classification of  $(S)$  on  $\mathbb{R}^2$  (or  $\mathcal{V}|_{\mathcal{H}}$ ) completed with its points "at infinity", i.e. on the equator  $S^1$  of  $S^2$ . Since the vertical projection is a diffeomorphism of  $\mathcal{H}'$  on the disk  $\{(x, y) | x^2 + y^2 \leq 1\}$  we can view the phase portraits of our systems  $(S)$  on this disk, called the Poincaré disk.

We shall also use the compactifications (real or complex) associated to the foliations with singularities (real or complex) attached to a real polynomial system  $(S)$  (cf. [8] or [25]). These foliations can be described as follows: For a real polynomial system  $(S)$  with  $n = \max(\deg(p), \deg(q))$  we associate to the two polynomials  $p, q \in \mathbb{R}[x, y]$  defining  $(S)$ , the homogeneous polynomials  $P, Q$  in  $X, Y, Z$ , of degree  $n$  with real coefficients, defined as follows:

$$P(X, Y, Z) = Z^n p(X/Z, Y/Z), \quad Q(X, Y, Z) = Z^n q(X/Z, Y/Z).$$

The real (respectively complex) foliations with singularities associated to  $(S)$  on the real projective plane  $\mathbb{P}^2(\mathbb{R})$  (respectively complex,  $\mathbb{P}^2(\mathbb{C})$ ) are then described in homogeneous coordinates by the equation

$$A(X, Y, Z)dX + B(X, Y, Z)dY + C(X, Y, Z)dZ = 0, \quad (2.1)$$

where  $A = ZQ$ ,  $B = -ZP$ ,  $C(X, Y, Z) = YP(X, Y, Z) - XQ(X, Y, Z)$  verify the following equality

$$A(X, Y, Z)X + B(X, Y, Z)Y + C(X, Y, Z)Z = 0 \quad (2.2)$$

in  $\mathbb{R}[X, Y, Z]$ . (For more details see [8] or [25]).

Our goal in this work is to give a topological classification, in terms of both geometric and algebraic invariants, of the quadratic systems  $(S)$  and their compactification on  $H'$  in the neighbourhood of the equator in the closed upper hemisphere  $H'$  of the Poincaré

sphere. Correspondingly this yields a topological classification of the real foliations, in the neighbourhood of the line at infinity associated to the imbedding of the affine plane:

$$j : \mathbf{A}^2(\mathbb{R}) = \mathbb{R}^2 \longrightarrow \mathbb{P}^2(\mathbb{R})$$

where  $j(x, y) = [x : y : 1]$ . The line at infinity in this case is therefore  $Z = 0$ .

### 3 Divisors on the line at infinity encoding globally the multiplicities of singularities

In this section we consider real polynomial systems  $(S)$  with  $n = \max(\deg(p), \deg(q))$  and their associated foliations with singularities, real or complex, defined in the previous section by the equation (2.1) with coefficients  $A, B, C$  verifying (2.2).

**Definition 3.1.** *For a system  $(S)$  we call divisor on the line at infinity, a formal expression of the form  $D = \sum n(w)w$  where  $w$  is a point of the complex line  $Z = 0$  of the complex projective plane,  $n(w)$  is an integer and only a finite number of the numbers  $n(w)$  are not zero. We call degree of the divisor  $D$  the integer  $\deg(D) = \sum n(w)$ . We call support of the divisor  $D$  the set  $\text{Supp}(D)$  of points  $w$  such that  $n(w) \neq 0$ .*

For systems  $(S)$  two divisors on the line at infinity were introduced in [26]. These were applied in [14] for classifying topologically the quadratic systems with a weak focus of third order.

**Definition 3.2.** *Assume that a system  $(S)$  is such that  $p(x, y)$  and  $q(x, y)$  are relatively prime over  $\mathbb{C}$  and that  $yp_n - xq_n$  is not identically zero (i.e.  $Z \nmid C$ ). Here  $p_n$  (respectively  $q_n$ ) is the sum of terms of degree  $n$  of  $p$  (respectively of  $q$ ) in case at least one of them has a non-zero coefficient and zero otherwise.*

*The following divisor on the line at infinity is then well defined:*

$$D_S(P, Q; Z) = \sum I_w(P, Q)w$$

*where the sum is taken for all points  $w = [X : Y : 0]$  on the line  $Z = 0$  and  $I_w(P, Q)$  is the intersection number (or multiplicity of intersection) at  $w$  (cf. [11]) of the complex projective curves  $P(X, Y, Z) = 0$  and  $Q(X, Y, Z) = 0$ .*

We thus have  $\text{Supp}(D_S(P, Q; Z)) = \{w \in \{Z = 0\} | P(w) = 0 = Q(w)\}$ .

The above divisor is a purely geometric object which encodes the contribution to the multiplicities of the singularities at infinity of the system  $(S)$ , arising from singularities in the finite plane, i.e. how many singular points in the finite plane could appear from those singularities at infinity in (polynomial) perturbations of  $(S)$ .

Let us list a number of integer-valued invariants which are attached to this divisor.

**Notation 3.1.**

$$\begin{aligned} N_{\infty,f}(S) &= \# \text{ Supp}(D_S(P, Q; Z)); \\ \nu(S) &= \max\{I_w(P, Q) \mid w \in \text{Supp}(D_S(P, Q; Z))\}; \\ \text{for every } m \leq \nu(S), \quad s(m) &= \# \{w \in \{Z = 0\} \mid I_w(P, Q) = m\} . \end{aligned}$$

Note that  $N_{\infty,f}$  is the number of distinct infinite singularities of  $(S)$  which could produce finite singular points in a (polynomial) perturbation of  $(S)$ .

We also need another divisor on the line at infinity which was used in [26] and [14] and which is defined as follows:

**Definition 3.3.** *Suppose  $Z \nmid C$  and consider*

$$D_S(C, Z) = \sum I_w(C, Z)w$$

*where the sum is taken for all points  $w = [X : Y : 0]$  on the line  $Z = 0$  of the complex projective plane.*

Clearly for quadratic differential systems  $\deg(D_S(C, Z)) = 3$ .

**Definition 3.4.** *A point  $w$  of the projective plane  $\mathbb{P}^2(\mathbb{C})$  is said to be of multiplicity  $(r, s)$  for a system  $(S)$  if*

$$(r, s) = (I_w(P, Q), I_w(C, Z)).$$

Following [26] we fuse the above two divisors on the line at infinity into just one but with values in the ring  $\mathbb{Z}^2$ :

**Definition 3.5.**

$$D_S = \sum \begin{pmatrix} I_w(P, Q) \\ I_w(C, Z) \end{pmatrix} w$$

*where  $w$  belongs to the line  $Z = 0$  of the complex projective plane.*



The above defined divisor describes the number of singularities which could arise in a perturbation of  $(S)$  from singularities at infinity of  $(S)$  in both the finite plane and at infinity.

**Definition 3.6.** *We call type of the divisor  $D_S(P, Q; Z)$  the set*

$$\{(s(m), m) \mid m \leq \nu(S)\}.$$

**Remark 3.1.** *We observe that the types of  $D_S(P, Q; Z)$  and of  $D_S(C, Z)$  are affine invariants since both  $I_w(P, Q)$  and  $I_w(C, Z)$  remain invariant under the action of the affine group on systems  $(S)$  ([19], [24]).*

**Notation 3.2.** *Let us introduce for planar systems  $(S)$  the following notations:*

$$\Delta_S = \deg D_S(P, Q; Z), \quad M_C = \max\{I_w(C, Z) \mid w \in \text{Supp}(D_S(C, Z))\}.$$

Consider a real quadratic differential system  $(S)$ :

$$\begin{aligned} \frac{dx}{dt} &= p_0 + p_1(x, y) + p_2(x, y) \equiv p(x, y), \\ \frac{dy}{dt} &= q_0 + q_1(x, y) + q_2(x, y) \equiv q(x, y). \end{aligned} \tag{3.1}$$

Suppose  $\gcd(p, q) = \text{constant}$ , where  $p_i$  (respectively  $q_i$ ) is the sum of terms in  $x$  and  $y$  of degree  $i$  of  $p$  (respectively of  $q$ ) in case at least one such term has non-zero coefficient and zero otherwise. Recall that **QS** denotes the class of all real quadratic systems.

We want to list all possible divisors  $D_S$  for quadratic systems  $(S)$  and characterize in terms of invariants and comitants the types of these divisors. This would make possible for any given system and in any chart the computation of the type of its divisor  $D_S$ . To do this we need to construct invariants and comitants with respect to group actions, which we do in the next section.

## 4 Group actions on quadratic systems (3.1) and invariants and comitants with respect to these actions

### 4.1 Group actions on quadratic systems (3.1)

More explicitly the systems (3.1) can be written in the form:

$$\begin{aligned}\frac{dx}{dt} &= a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\ \frac{dy}{dt} &= b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + 2b_{11}xy + b_{02}y^2,\end{aligned}$$

and let  $a = (a_{00}, \dots, b_{02})$ . Consider the ring  $\mathbb{R}[a_{00}, a_{10}, \dots, a_{02}, b_{00}, b_{10}, \dots, b_{02}, x, y]$  which we shall denote  $\mathbb{R}[a, x, y]$ .

On the set **QS** of all quadratic differential systems (3.1) acts the group  $Aff(2, \mathbb{R})$  of affine transformations on the plane. Indeed for every  $g \in Aff(2, \mathbb{R})$ ,  $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$  we have:

$$g : \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix} + B; \quad g^{-1} : \begin{pmatrix} x \\ y \end{pmatrix} = M^{-1} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} - M^{-1}B.$$

where  $M = ||M_{ij}||$  is a  $2 \times 2$  nonsingular matrix and  $B$  is a  $2 \times 1$  matrix over  $\mathbb{R}$ . For every  $S \in \mathbf{QS}$  we can form its transformed system  $\tilde{S} = gS$ :

$$\frac{d\tilde{x}}{dt} = \tilde{p}(\tilde{x}, \tilde{y}), \quad \frac{d\tilde{y}}{dt} = \tilde{q}(\tilde{x}, \tilde{y}), \quad (\tilde{S})$$

where

$$\begin{pmatrix} \tilde{p}(\tilde{x}, \tilde{y}) \\ \tilde{q}(\tilde{x}, \tilde{y}) \end{pmatrix} = M \begin{pmatrix} (p \circ g^{-1})(\tilde{x}, \tilde{y}) \\ (q \circ g^{-1})(\tilde{x}, \tilde{y}) \end{pmatrix}.$$

The map

$$\begin{aligned}Aff(2, \mathbb{R}) \times \mathbf{QS} &\longrightarrow \mathbf{QS} \\ (g, S) &\longrightarrow \tilde{S} = gS\end{aligned}$$

verifies the axioms for a left group action. For every subgroup  $G \subseteq Aff(2, \mathbb{R})$  we have an induced action of  $G$  on **QS**. We can identify the set **QS** of system (3.1) with a subset of  $\mathbb{R}^{12}$  via the embedding  $\mathbf{QS} \hookrightarrow \mathbb{R}^{12}$  which associates to each system (3.1) the 12-tuple  $(a_{00}, \dots, b_{02})$  of its coefficients.

On systems  $(S)$  such that  $\max(\deg(p), \deg(q)) \leq 2$  we consider the action of the group  $Aff(2, \mathbb{R})$  which yields an action of this group on  $\mathbb{R}^{12}$ . For every  $g \in Aff(2, \mathbb{R})$  let  $r_g :$

$\mathbb{R}^{12} \longrightarrow \mathbb{R}^{12}$  be the map which corresponds to  $g$  via this action. We know (cf. [30]) that  $r_g$  is linear and that the map  $r : \text{Aff}(2, \mathbb{R}) \longrightarrow GL(12, \mathbb{R})$  thus obtained is a group homomorphism. For every subgroup  $G$  of  $\text{Aff}(2, \mathbb{R})$ ,  $r$  induces a representation of  $G$  onto a subgroup  $\mathcal{G}$  of  $GL(12, \mathbb{R})$ .

## 4.2 Invariants and comitants associated to the group actions

**Definition 4.1.** A polynomial  $U(a, x, y) \in \mathbb{R}[a, x, y]$  is called a comitant of systems (3.1) with respect to a subgroup  $G$  of  $\text{Aff}(2, \mathbb{R})$ , if there exists  $\chi \in \mathbb{Z}$  such that for every  $(g, \mathbf{a}) \in G \times \mathbb{R}^{12}$  and for every  $(x, y) \in \mathbb{R}^2$  the following relation holds:

$$U(r_g(\mathbf{a}), g(x, y)) \equiv (\det g)^{-\chi} U(\mathbf{a}, x, y),$$

where  $\det g = \det M$ . If the polynomial  $U$  does not explicitly depend on  $x$  and  $y$  then it is called invariant. The number  $\chi \in \mathbb{Z}$  is called the weight of the comitant  $U(a, x, y)$ . If  $G = GL(2, \mathbb{R})$  (or  $G = \text{Aff}(2, \mathbb{R})$ ) then the comitant  $U(a, x, y)$  of systems (3.1) is called  $GL$ -comitant (respectively, affine comitant).

**Definition 4.2.** A subset  $X \subset \mathbb{R}^{12}$  will be called  $G$ -invariant, if for every  $g \in G$  we have  $r_g(X) \subseteq X$ .

As it can easily be verified, the following polynomials are  $GL$ -comitants of system (3.1):

$$\begin{aligned} C_i(a, x, y) &= yp_i(x, y) - xq_i(x, y), \quad i = 0, 1, 2; \\ M(a, x, y) &= 2 \text{Hess} (C_2(a, x, y)); \\ \eta(a) &= \text{Discrim} (C_2(a, x, y)); \\ K(a, x, y) &= \text{Jacob} (p_2(x, y), q_2(x, y)); \\ \mu_0(a) &= \text{Res}_x(p_2, q_2)/y^4 = \text{Discrim} (K(a, x, y))/16; \\ H(a, x, y) &= - \text{Discrim} (\alpha p_2(x, y) + \beta q_2(x, y))|_{\{\alpha=y, \beta=-x\}}; \\ L(a, x, y) &= 2K - 4H - M; \\ K_1(a, x, y) &= p_1(x, y)q_2(x, y) - p_2(x, y)q_1(x, y). \end{aligned} \tag{4.1}$$

Let  $T(2, \mathbb{R})$  be the subgroup of  $\text{Aff}(2, \mathbb{R})$  formed by translations. Consider the linear representation of  $T(2, \mathbb{R})$  into its corresponding subgroup  $\mathcal{T} \subset GL(12, \mathbb{R})$ , i.e. for every  $\tau \in T(2, \mathbb{R})$ ,  $\tau : x = \tilde{x} + \alpha, y = \tilde{y} + \beta$  we consider as above  $r_\tau : \mathbb{R}^{12} \longrightarrow \mathbb{R}^{12}$ .

**Definition 4.3.** A  $GL$ -comitant  $U(a, x, y)$  of systems (3.1) is called a  $T$ -comitant if for every  $(\tau, \mathbf{a}) \in T(2, \mathbb{R}) \times \mathbb{R}^{12}$  and for every  $(\tilde{x}, \tilde{y}) \in \mathbb{R}^2$  the relation  $U(r_\tau \cdot \mathbf{a}, \tilde{x}, \tilde{y}) = U(\mathbf{a}, \tilde{x}, \tilde{y})$  holds.

Let

$$U_i(a, x, y) = \sum_{j=0}^{d_i} U_{ij}(a) x^{d_i-j} y^j, \quad i = 1, \dots, s$$

be a set of  $GL$ -comitants of systems (3.1) where  $d_i$  denotes the degree of the binary form  $U_i(a, x, y)$  in  $x$  and  $y$  with coefficients in  $\mathbb{R}[a]$  where  $\mathbb{R}[a] = \mathbb{R}[a_{00}, \dots, b_{02}]$ . We denote by

$$\mathcal{U} = \{ U_{ij}(a) \in \mathbb{R}[a] \mid i = 1, \dots, s, j = 0, 1, \dots, d_i \},$$

the set of the coefficients in  $\mathbb{R}[a]$  of the  $GL$ -comitants  $U_i(a, x, y)$ ,  $i = 1, \dots, s$ , and by  $V(\mathcal{U})$  its associated algebraic set:

$$V(\mathcal{U}) = \{ \mathbf{a} \in \mathbb{R}^{12} \mid U_{ij}(\mathbf{a}) = 0 \ \forall \ U_{ij}(a) \in \mathcal{U} \}.$$

**Definition 4.4.** A  $GL$ -comitant  $U(a, x, y)$  of systems (3.1) is called a conditional  $T$ -comitant (or  $CT$ -comitant) modulo  $\langle U_1, U_2, \dots, U_s \rangle$  if the following two conditions are satisfied:

- (i) the algebraic subset  $V(\mathcal{U}) \subset \mathbb{R}^{12}$  is affinely invariant (see Definition 4.2);
- (ii) for every  $(\tau, \mathbf{a}) \in T(2, \mathbb{R}) \times V(\mathcal{U})$  we have  $U(r_\tau \cdot \mathbf{a}, \tilde{x}, \tilde{y}) = U(\mathbf{a}, \tilde{x}, \tilde{y})$  in  $\mathbb{R}[\tilde{x}, \tilde{y}]$ .

In other words, a  $CT$ -comitant  $U(a, x, y)$  modulo  $\langle U_1, U_2, \dots, U_s \rangle$  is a  $T$ -comitant on the algebraic subset  $V(\mathcal{U}) \subset \mathbb{R}^{12}$ .

**Proposition 4.1.** Let  $S \in \mathbf{QS}$  and let  $\mathbf{a} \in \mathbb{R}^{12}$  be its 12-tuple of coefficients. The common points of  $P = 0$  and  $Q = 0$  on the line  $Z = 0$  are given by the common linear factors over  $\mathbb{C}$  of  $p_2$  and  $q_2$ . This yields the geometrical meaning of the comitants  $\mu_0$ ,  $K$  and  $H$ :

$$\gcd(p_2(x, y), q_2(x, y)) = \begin{cases} \text{constant} & \text{iff } \mu_0(\mathbf{a}) \neq 0; \\ bx + cy & \text{iff } \mu_0 = 0, K(\mathbf{a}, x, y) \neq 0; \\ (bx + cy)(dx + ey) & \text{iff } \begin{cases} \mu_0(\mathbf{a}) = 0, K(\mathbf{a}, x, y) = 0 \\ \text{and } H(\mathbf{a}, x, y) \neq 0; \end{cases} \\ (bx + cy)^2 & \text{iff } \begin{cases} \mu_0 = 0, K(\mathbf{a}, x, y) = 0, \\ \text{and } H(\mathbf{a}, x, y) = 0; \end{cases} \end{cases}$$

where  $bx + cy, dx + ey \in \mathbb{C}[x, y]$  are some linear forms and  $be - cd \neq 0$ .

**Definition 4.5.** *The polynomial  $U(a, x, y) \in \mathbb{R}[a, x, y]$  has well determined sign on  $V \subset \mathbb{R}^{12}$  with respect to  $x, y$  if for every fixed  $\mathbf{a} \in V$ , the sign of the polynomial function  $U(\mathbf{a}, x, y)$  on  $\mathbb{R}^2$  is constant where this function is not zero.*

**Observation 4.1.** *We draw the attention to the fact, that if a  $CT$ -comitant  $U(a, x, y)$  of even weight is a binary form in  $x, y$ , of even degree in the coefficients of (3.1) and has well determined sign on some affine invariant algebraic subset  $V(\mathcal{U})$  then this property is conserved by any affine transformation and the sign is conserved.*

### 4.3 Canonical forms of planar quadratic systems in the neighbourhood of infinity

**Lemma 4.1.** *For a system (3.1) with  $C_2(\mathbf{a}, x, y) \not\equiv 0$  the divisor  $D_S(C, Z)$  is well defined and its type is determined by the corresponding conditions indicated in Table 1, where we write  $q_1^c + q_2^c + q_3$  if two of the points, i.e.  $q_1^c, q_2^c$ , are complex but not real. Moreover, for each type of the divisor  $D_S(C, Z)$  given by Table 1 the quadratic systems (3.1) can be brought via a linear transformation to one of the following canonical systems  $(\mathbf{S}_I) - (\mathbf{S}_{IV})$  corresponding to their behavior at infinity.*

*Proof:* The Table 1 follows easily from the definitions of  $\eta(a)$  and  $M(a, x, y)$  in (4.1). Let us consider the  $GL$ -comitant  $C_2(a, x, y) \not\equiv 0$  simply as a cubic binary form in  $x$  and  $y$ . For every  $\mathbf{a} \in \mathbb{R}^{12}$  the binary form  $C_2(\mathbf{a}, x, y)$  can be reduced to one of the canonical forms given below, by a linear transformation, i.e. there exist  $g \in GL(2, \mathbb{R})$ :  $g(x, y) = (u, v)$  such that the transformed binary form  $gC_2(\mathbf{a}, x, y) = C_2(\mathbf{a}, g^{-1}(u, v))$  is one of the following

$$I. xy(x - y); \quad II. x(x^2 + y^2); \quad III. x^2y; \quad IV. x^3, \quad (4.2)$$

which correspond to the types of the divisor  $D_S(C, Z)$  indicated in Table 1. On the other hand, according to the Definition 4.1 of the  $GL$ -comitant, for  $C_2(a, x, y)$  whose weight  $\chi = -1$ , we have for  $g \in GL(2, \mathbb{R})$

$$C_2(r_g(\mathbf{a}), g(x, y)) = \det(g) C_2(\mathbf{a}, x, y).$$

Using  $g(x, y) = (u, v)$  we obtain

$$C_2(r_g(\mathbf{a}), u, v) = \lambda C_2(\mathbf{a}, g^{-1}(u, v)), \quad \lambda \in \mathbb{R},$$

where we may consider  $\lambda = 1$  by rescaling :  $u = u_1/\lambda, v = v_1/\lambda$ .

**Table 1**

$M_C$	Type of $D_S(C, Z)$	Necessary and sufficient conditions on the comitants	Notation for the conditions
1	$q_1 + q_2 + q_3$	$\eta > 0$	$(\mathcal{I}_1)$
	$q_1^c + q_2^c + q_3$	$\eta < 0$	$(\mathcal{I}_2)$
2	$2q_1 + q_2$	$\eta = 0, \quad M \neq 0$	$(\mathcal{I}_3)$
3	$3q$	$M = 0$	$(\mathcal{I}_4)$

$$\begin{cases} \frac{dx}{dt} = k + cx + dy + gx^2 + (h-1)xy, \\ \frac{dy}{dt} = l + ex + fy + (g-1)xy + hy^2; \end{cases} \quad (\mathbf{S}_I)$$

$$\begin{cases} \frac{dx}{dt} = k + cx + dy + gx^2 + (h+1)xy, \\ \frac{dy}{dt} = l + ex + fy - x^2 + gxy + hy^2; \end{cases} \quad (\mathbf{S}_{II})$$

$$\begin{cases} \frac{dx}{dt} = k + cx + dy + gx^2 + hxy, \\ \frac{dy}{dt} = l + ex + fy + (g-1)xy + hy^2; \end{cases} \quad (\mathbf{S}_{III})$$

$$\begin{cases} \frac{dx}{dt} = k + cx + dy + gx^2 + hxy, \\ \frac{dy}{dt} = l + ex + fy - x^2 + gxy + hy^2, \end{cases} \quad (\mathbf{S}_{IV})$$

Thus, recalling that

$$p_2(x, y) = a_{20}x^2 + 2a_{11}x, y + a_{02}y^2, \quad q_2(x, y) = b_{20}x^2 + 2b_{11}x, y + b_{02}y^2,$$

for the first canonical form in (4.2) we have

$$C_2(\mathbf{a}, x, y) = -b_{20}x^3 + (a_{20} - 2b_{11})x^2y + (2a_{11} - b_{02})xy^2 + a_{02}y^3 = xy(x - y).$$

Identifying the coefficients of the above identity we get the canonical form  $(\mathbf{S}_I)$ : Analogously for the cases  $II$ ,  $III$  and  $IV$  we obtain the canonical form  $(\mathbf{S}_{II})$ ,  $(\mathbf{S}_{III})$  and  $(\mathbf{S}_{IV})$  associated to the respective polynomials in (4.2). ■

## 5 Classification of the quadratic systems according to the types of the multiplicity divisor $\mathcal{D}_S$

A specific type of a divisor  $D_S$  yields a class of quadratic systems (3.1). We want to list all possible types of the divisors  $D_S$  and for each specific type to determine the subset of **QS** where  $D_S$  has this type. We want to give this subset in terms of algebraic invariants and comitants so as to be able to check these conditions for every system (3.1) in any chart.

In order to construct other necessary invariant polynomials let us consider the differential operator  $\mathcal{L} = x \cdot \mathbf{L}_2 - y \cdot \mathbf{L}_1$  acting on  $\mathbb{R}[a, x, y]$  constructed in [5], where

$$\begin{aligned}\mathbf{L}_1 &= 2a_{00} \frac{\partial}{\partial a_{10}} + a_{10} \frac{\partial}{\partial a_{20}} + \frac{1}{2}a_{01} \frac{\partial}{\partial a_{11}} + 2b_{00} \frac{\partial}{\partial b_{10}} + b_{10} \frac{\partial}{\partial b_{20}} + \frac{1}{2}b_{01} \frac{\partial}{\partial b_{11}}; \\ \mathbf{L}_2 &= 2a_{00} \frac{\partial}{\partial a_{01}} + a_{01} \frac{\partial}{\partial a_{02}} + \frac{1}{2}a_{10} \frac{\partial}{\partial a_{11}} + 2b_{00} \frac{\partial}{\partial b_{01}} + b_{01} \frac{\partial}{\partial b_{02}} + \frac{1}{2}b_{10} \frac{\partial}{\partial b_{11}}\end{aligned}$$

as well as the classical differential operator  $(f, \varphi)^{(2)}$  acting on  $\mathbb{R}[a, x, y]$  which is called *transvectant* of the second index (see, for example, [12, 17]):

$$(f, \varphi)^{(2)} = \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 \varphi}{\partial y^2} - 2 \frac{\partial^2 f}{\partial x \partial y} \frac{\partial^2 \varphi}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} \frac{\partial^2 \varphi}{\partial x^2}.$$

Here  $f(x, y)$  and  $\varphi(x, y)$  are polynomials in  $x$  and  $y$ .

In [5] it is shown that if a polynomial  $U \in \mathbb{R}[a, x, y]$  is a comitant of system (3.1) with respect to the group  $GL(2, \mathbb{R})$  then  $\mathcal{L}(U)$  is also a  $GL$ -comitant. The same is true for the operator transvectant of two comitants,  $f$  and  $\varphi$ .

So, by using these operators and the  $GL$ -comitants  $\mu_0(a)$ ,  $M(a, x, y)$  and  $K(a, x, y)$  we shall construct the following polynomials:

$$\mu_i(a, x, y) = \frac{1}{i!} \mathcal{L}^{(i)}(\mu_0), \quad i = 1, \dots, 4, \quad \kappa(a) = (M, K)^{(2)}, \quad \kappa_1(a) = (M, C_1)^{(2)}, \quad (5.1)$$

where  $\mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0))$ .

These polynomials are in fact comitants of system (3.1) with respect to the group  $GL(2, \mathbb{R})$ .

To reveal the geometrical meaning of the comitants  $\mu_i(a, x, y)$ ,  $i = 0, 1, \dots, 4$  we use the following resultants whose calculation yield:

$$Res_x(P, Q) = \mu_0 Y^4 + \mu_{10} Y^3 Z + \mu_{20} Y^2 Z^2 + \mu_{30} Y Z^3 + \mu_{40} Z^4; \quad (5.2)$$

$$Res_y(P, Q) = \mu_0 X^4 + \mu_{01} X^3 Z + \mu_{02} X^2 Z^2 + \mu_{03} X Z^3 + \mu_{04} Z^4, \quad (5.3)$$

where  $\mu_{ij} = \mu_{ij}(a) \in \mathbb{R}[a_{00}, \dots, b_{02}]$ .

On the other hand for  $\mu_i$ ,  $i = 0, 1, \dots, 4$  from (5.1) we have

$$\begin{aligned}\mu_0(a) &= \mu_0; \\ \mu_1(a, x, y) &= \mu_{10}x + \mu_{01}y; \\ \mu_2(a, x, y) &= \mu_{20}x^2 + \mu_{11}xy + \mu_{02}y^2; \\ \mu_3(a, x, y) &= \mu_{30}x^3 + \mu_{21}x^2y + \mu_{12}xy^2 + \mu_{03}y^3; \\ \mu_4(a, x, y) &= \mu_{40}x^4 + \mu_{31}x^3y + \mu_{22}x^2y^2 + \mu_{13}xy^3 + \mu_{04}y^4.\end{aligned}$$

We observe that the leading coefficients of the comitants  $\mu_i$ ,  $i = 0, 1, \dots, 4$  with respect to  $x$  (respectively  $y$ ) are the corresponding coefficients in (5.2) (respectively (5.3)).

We draw the attention to the fact, that if the comitant  $\mu_i(a, x, y)$  ( $i = 0, 1, \dots, 4$ ) is not equal to zero then we may assume that its leading coefficients are both non zero, as this can be obtained by applying a rotation of the phase plane of the system (3.1). From here and (5.2), (5.3) and the above values of  $\mu_i$ ,  $i = 0, 1, \dots, 4$  we have:

**Lemma 5.1.** *The system  $P(X, Y, Z) = Q(X, Y, Z) = 0$  possesses  $m$  ( $= \Delta_S$ ) ( $1 \leq m \leq 4$ ) solutions  $[X_i : Y_i : Z_i]$  with  $Z_i = 0$  ( $i = 1, \dots, m$ ) (considered with multiplicities) if and only if for every  $i \in \{0, 1, \dots, m-1\}$  we have  $\mu_i(a, x, y) = 0$  in  $\mathbb{R}[a, x, y]$  and  $\mu_m(a, x, y) \neq 0$ .*

**Remark 5.1.** *It can easily be checked that the following identity holds*

$$\mu_4(a, X, Y) = \text{Res}_Z(P(X, Y, Z), Q(X, Y, Z)).$$

Hence, clearly for any solution  $[X_0 : Y_0 : Z_0]$  (including those with  $Z_0 = 0$ ) of the system of equations  $P(X, Y, Z) = Q(X, Y, Z) = 0$ , the following relation is satisfied:  $\mu_4(a, X_0, Y_0) = 0$ .

We give below our theorem of classification of the types of all divisors  $D_S$  occurring in quadratic systems and associate to each type the necessary and sufficient conditions in terms of algebraic invariants and comitants. The computation of these invariants and comitants can be programmed using symbolic manipulations and implemented on computers. Thus for any specific system (3.1) we can calculate explicitly its divisor type in whatever chart (3.1) is given.

**Theorem 5.1.** *We consider here the family  $\mathbf{QS}_{\text{ess}}$  of all systems  $(S)$  in  $\mathbf{QS}$  which are essentially quadratic, i.e.  $\gcd(P, Q) = 1$  and  $Z \nmid C$ . All possible values which could be taken by  $\Delta_S$  for such systems (3.1) are as listed in the first column of Table 2. For each value of  $\Delta_S$ , all possibilities we have for  $M_C$ , are listed in the second column. For each*



combination  $(\Delta_S, M_C)$  all the possibilities we have for the form of  $D_S$  are those indicated in the third column. For a specified  $(\Delta_S, M_C)$ , the necessary and sufficient conditions to have the form of  $D_S$  as indicated in the third column are those indicated in the corresponding fourth column. (We recall that  $\mathcal{I}_j$  are the conditions indicated in Table 1. In the last column of Table 2 we denote by  $\Sigma_i$  the class of all quadratic systems which possess  $(\Delta_S, M_C, D_S)$  as indicated in the first three columns).

*Proof:* We need to examine the four distinct cases corresponding to the canonical forms  $(\mathbf{S}_I) - (\mathbf{S}_IV)$ , respectively.

## 5.1 Systems of type $\mathbf{S}_I$

For systems  $(\mathbf{S}_I)$  we have  $\mu_0 = gh(g + h - 1)$  and for  $\mu_0 \neq 0$  according to Lemma 5.1 we have  $\Delta_S = 0$  and, hence, we obtain a system of the class  $\Sigma_1$  (see Table 2).

Let us consider now  $\mu_0 = 0$ . In this case we have  $gh(g + h - 1) = 0$  and without loss of generality we may assume  $g = 0$ . Indeed, if  $h = 0$  (respectively,  $g + h - 1 = 0$ ) we can apply the linear transformation which will replace the straight line  $y = 0$  with  $x = 0$  (respectively,  $y = 0$  with  $y = x$ ). Let  $g = 0$ . By using the translation  $x = x_1 + (f + eh)/2$ ,  $y = y_1 + e/2$  we may assume  $e = f = 0$ . In this way the system  $(\mathbf{S}_I)$  will be brought to the following canonical form:

$$\dot{x} = k + cx + dy + (h - 1)xy, \quad \dot{y} = l - xy + hy^2, \quad (5.4)$$

for which we have

$$\mu_1 = ch(1 - h)y, \quad \kappa = 64h(1 - h), \quad K = 2h(h - 1)y^2.$$

For  $\mu_1 \neq 0$ , from Lemma 5.1 we obtain  $\Delta_S = 1$  which leads us to the case  $\Sigma_5$ .

Considering  $\mu_1 = 0$  we shall examine two cases:  $\kappa \neq 0$  and  $\kappa = 0$ .

### 5.1.1 Case $\kappa \neq 0$

As the condition  $\kappa \neq 0$  is equivalent to condition  $K \neq 0$ , according to Proposition 4.1 we conclude that  $\text{Supp } D_S(P, Q; Z)$  contains exactly one point  $p = [1 : 0 : 0]$  since  $\gcd(p_2, q_2) = y$ . By Lemma 5.1 its multiplicity  $I_p(P, Q)$  depends of the number of vanishing comitants  $\mu_i(a, x, y)$ . In this way we obtain that a quadratic system belongs to the set  $\Sigma_{10}$  (respectively  $\Sigma_{18}$ ;  $\Sigma_{26}$ ) for  $\mu_{0,1} = 0$ ,  $\mu_2 \neq 0$  (respectively for  $\mu_{0,1,2} = 0$ ,  $\mu_3 \neq 0$ ;  $\mu_{0,1,2,3} = 0$ ,  $\mu_4 \neq 0$ ). We use the compact notation  $\mu_{0,1,2} = 0$  for  $\mu_0 = \mu_1 = \mu_2 = 0$ .

**Table 2**

$\Delta_S$	$M_C$	Value of $D_S$	Necessary and sufficient conditions on the comitants	$\Sigma_i$
0	1	$\binom{0}{1}p + \binom{0}{1}q + \binom{0}{1}r$	$\mu_0 \neq 0, (\mathcal{I}_1)$	$\Sigma_1$
		$\binom{0}{1}p + \binom{0}{1}q^c + \binom{0}{1}r^c$	$\mu_0 \neq 0, (\mathcal{I}_2)$	$\Sigma_2$
	2	$\binom{0}{1}p + \binom{0}{2}q$	$\mu_0 \neq 0, (\mathcal{I}_3)$	$\Sigma_3$
	3	$\binom{0}{3}p$	$\mu_0 \neq 0, (\mathcal{I}_4)$	$\Sigma_4$
1	1	$\binom{1}{1}p + \binom{0}{1}q + \binom{0}{1}r$	$\mu_0 = 0, \mu_1 \neq 0, (\mathcal{I}_1)$	$\Sigma_5$
		$\binom{1}{1}p + \binom{0}{1}q^c + \binom{0}{1}r^c$	$\mu_0 = 0, \mu_1 \neq 0, (\mathcal{I}_2)$	$\Sigma_6$
	2	$\binom{1}{1}p + \binom{0}{2}q$	$\mu_0 = 0, \mu_1 \neq 0, \kappa \neq 0, (\mathcal{I}_3)$	$\Sigma_7$
		$\binom{0}{1}p + \binom{1}{2}q$	$\mu_0 = 0, \mu_1 \neq 0, \kappa = 0, (\mathcal{I}_3)$	$\Sigma_8$
	3	$\binom{1}{3}p$	$\mu_0 = 0, \mu_1 \neq 0, (\mathcal{I}_4)$	$\Sigma_9$
2	1	$\binom{2}{1}p + \binom{0}{1}q + \binom{0}{1}r$	$\mu_{0,1} = 0, \mu_2 \neq 0, \kappa \neq 0, (\mathcal{I}_1)$	$\Sigma_{10}$
		$\binom{1}{1}p + \binom{1}{1}q + \binom{0}{1}r$	$\mu_{0,1} = 0, \mu_2 \neq 0, \kappa = 0, (\mathcal{I}_1)$	$\Sigma_{11}$
		$\binom{2}{1}p + \binom{0}{1}q^c + \binom{0}{1}r^c$	$\mu_{0,1} = 0, \mu_2 \neq 0, \kappa \neq 0, (\mathcal{I}_2)$	$\Sigma_{12}$
		$\binom{0}{1}p + \binom{1}{1}q^c + \binom{1}{1}r^c$	$\mu_{0,1} = 0, \mu_2 \neq 0, \kappa = 0, (\mathcal{I}_2)$	$\Sigma_{13}$
	2	$\binom{2}{1}p + \binom{0}{2}q$	$\mu_{0,1} = 0, \mu_2 \neq 0, \kappa \neq 0, (\mathcal{I}_3)$	$\Sigma_{14}$
		$\binom{1}{1}p + \binom{1}{2}q$	$\mu_{0,1} = 0, \mu_2 \neq 0, \kappa = 0, L = 0, (\mathcal{I}_3)$	$\Sigma_{15}$
		$\binom{0}{1}p + \binom{2}{2}q$	$\mu_{0,1} = 0, \mu_2 \neq 0, \kappa = 0, L \neq 0, (\mathcal{I}_3)$	$\Sigma_{16}$
	3	$\binom{2}{3}p$	$\mu_{0,1} = 0, \mu_2 \neq 0, (\mathcal{I}_4)$	$\Sigma_{17}$
3	1	$\binom{3}{1}p + \binom{0}{1}q + \binom{0}{1}r$	$\mu_{0,1,2} = 0, \mu_3 \neq 0, \kappa \neq 0, (\mathcal{I}_1)$	$\Sigma_{18}$
		$\binom{2}{1}p + \binom{1}{1}q + \binom{0}{1}r$	$\mu_{0,1,2} = 0, \mu_3 \neq 0, \kappa = 0, (\mathcal{I}_1)$	$\Sigma_{19}$
		$\binom{3}{1}p + \binom{0}{1}q^c + \binom{0}{1}r^c$	$\mu_{0,1,2} = 0, \mu_3 \neq 0, (\mathcal{I}_2)$	$\Sigma_{20}$
	2	$\binom{3}{1}p + \binom{0}{2}q$	$\mu_{0,1,2} = 0, \mu_3 \neq 0, \kappa \neq 0, (\mathcal{I}_3)$	$\Sigma_{21}$
		$\binom{2}{1}p + \binom{1}{2}q$	$\mu_{0,1,2} = 0, \mu_3 \neq 0, \kappa = L = 0, \kappa_1 \neq 0, (\mathcal{I}_3)$	$\Sigma_{22}$
		$\binom{1}{1}p + \binom{2}{2}q$	$\mu_{0,1,2} = 0, \mu_3 \neq 0, \kappa = L = 0, \kappa_1 = 0, (\mathcal{I}_3)$	$\Sigma_{23}$
		$\binom{0}{1}p + \binom{3}{2}q$	$\mu_{0,1,2} = 0, \mu_3 \neq 0, \kappa = 0, L \neq 0, (\mathcal{I}_3)$	$\Sigma_{24}$
	3	$\binom{3}{3}p$	$\mu_{0,1,2} = 0, \mu_3 \neq 0, (\mathcal{I}_4)$	$\Sigma_{25}$

**Table 2** (continued)

$\Delta_S$	$M_C$	Value of $D_S$	Necessary and sufficient conditions on the comitants	$\Sigma_i$
4	1	$\binom{4}{1}p + \binom{0}{1}q + \binom{0}{1}r$	$\mu_{0,1,2,3}=0, \mu_4 \neq 0, \kappa \neq 0, (\mathcal{I}_1)$	$\Sigma_{26}$
		$\binom{3}{1}p + \binom{1}{1}q + \binom{0}{1}r$	$\mu_{0,1,2,3}=0, \mu_4 \neq 0, \kappa = 0, K_1 \neq 0, (\mathcal{I}_1)$	$\Sigma_{27}$
		$\binom{2}{1}p + \binom{2}{1}q + \binom{0}{1}r$	$\mu_{0,1,2,3}=0, \mu_4 \neq 0, \kappa = 0, K_1 = 0, (\mathcal{I}_1)$	$\Sigma_{28}$
		$\binom{4}{1}p + \binom{0}{1}q^c + \binom{0}{1}r^c$	$\mu_{0,1,2,3}=0, \mu_4 \neq 0, \kappa \neq 0, (\mathcal{I}_2)$	$\Sigma_{29}$
		$\binom{0}{1}p + \binom{2}{1}q^c + \binom{2}{1}r^c$	$\mu_{0,1,2,3}=0, \mu_4 \neq 0, \kappa = 0, (\mathcal{I}_2)$	$\Sigma_{30}$
	2	$\binom{4}{1}p + \binom{0}{2}q$	$\mu_{0,1,2,3}=0, \mu_4 \neq 0, \kappa \neq 0, (\mathcal{I}_3)$	$\Sigma_{31}$
		$\binom{3}{1}p + \binom{1}{2}q$	$\mu_{0,1,2,3}=0, \mu_4 \neq 0, \kappa = L = 0, \kappa_1 \neq 0, (\mathcal{I}_3)$	$\Sigma_{32}$
		$\binom{2}{1}p + \binom{2}{2}q$	$\mu_{0,1,2,3}=0, \mu_4 \neq 0, \kappa = L = \kappa_1 = 0, K_1 = 0, (\mathcal{I}_3)$	$\Sigma_{33}$
		$\binom{1}{1}p + \binom{3}{2}q$	$\mu_{0,1,2,3}=0, \mu_4 \neq 0, \kappa = L = \kappa_1 = 0, K_1 \neq 0, (\mathcal{I}_3)$	$\Sigma_{34}$
		$\binom{0}{1}p + \binom{4}{2}q$	$\mu_{0,1,2,3}=0, \mu_4 \neq 0, \kappa = 0, L \neq 0, (\mathcal{I}_3)$	$\Sigma_{35}$
	3	$\binom{4}{3}p$	$\mu_{0,1,2,3} = 0, \mu_4 \neq 0, (\mathcal{I}_4)$	$\Sigma_{36}$

### 5.1.2 Case $\kappa = 0$

In this case  $h(h-1) = 0$  and analogously to the previous case, without loss of the generality we may assume  $h = 0$ . Thus, for system (5.4) we obtain:

$$\begin{aligned}\mu_0 = \mu_1 = 0, \quad \mu_2 = -cdxy, \quad \mu_3 = (k-l)(dy-cx)xy, \\ \mu_4 = -xy[lc^2x^2 - (k-l)^2xy + 2lcdxy + ld^2y^2], \quad K_1 = -xy(cx+dy).\end{aligned}$$

So, if  $\mu_2 \neq 0$  taking into consideration Remark 5.1 and the value of the comitant  $\mu_4$ , we obtain the case  $\Sigma_{11}$  in Table 2.

If  $\mu_2 = 0$  and  $\mu_3 \neq 0$  then  $cd = 0$ ,  $c^2 + d^2 \neq 0$  and clearly we arrive at the case  $\Sigma_{19}$ .

Let us now suppose that the conditions  $\mu_2 = \mu_3 = 0$  hold.

**5.1.2.1**  $K_1 \neq 0$ . Then  $c^2 + d^2 \neq 0$  and from  $\mu_3 = 0$  we obtain  $k = l$  which yields either  $\mu_4 = -ld^2xy^3$  (for  $c = 0$ ) or  $\mu_4 = -lc^2x^3y$  (for  $d = 0$ ). Both these cases lead us to the case  $\Sigma_{27}$  in Table 2.

**5.1.2.2**  $K_1 = 0$ . In this case it follows at once that  $c = d = 0$  and, hence,  $\mu_4 = 4(k-l)^2x^2y^2$ . Thus taking into consideration Remark 5.1 we obtain the case  $\Sigma_{28}$ .

## 5.2 Systems of type $(S_{II})$

For a canonical system  $(S_{II})$  we obtain

$$\begin{aligned}\mu_0 = -h[g^2 + (h+1)^2], \quad \kappa = -64[g^2 + (h+1)(1-3h)], \\ K = 2(g^2 + h+1)x^2 + 4ghxy + 2h(h+1)y^2\end{aligned}$$

and for  $\mu_0 \neq 0$  according to Lemma 5.1 we have  $\Delta_S = 0$ . Thus we obtain the case  $\Sigma_2$  in Table 2.

Let us consider now  $\mu_0 = 0$ , i.e.  $h[g^2 + (h+1)^2] = 0$ .

### 5.2.1 Case $\kappa \neq 0$

In this case we have  $h = 0$  and since the condition  $\kappa \neq 0$  is equivalent to the condition  $K \neq 0$ , according to Proposition 4.1,  $\text{Supp } D_S(P, Q; Z)$  contains only one point, namely the real one. By Lemma 5.1 its multiplicity depends of the number of the vanishing comitants  $\mu_i$ . Therefore the quadratic system belongs to the set  $\Sigma_6$  (respectively  $\Sigma_{12}$ ;  $\Sigma_{20}$ ;  $\Sigma_{29}$ ) for  $\mu_1 \neq 0$  (respectively for  $\mu_1 = 0, \mu_2 \neq 0$ ;  $\mu_{1,2} = 0, \mu_3 \neq 0$ ;  $\mu_{1,2,3} = 0, \mu_4 \neq 0$ ).

### 5.2.2 Case $\kappa = 0$

The conditions  $\mu_0 = \kappa = 0$  yield  $g = 0$ ,  $h = -1$  and translating the origin of coordinates at the point  $(e/4, f/4)$  the system  $(\mathbf{S}_{II})$  will be brought to the form

$$\dot{x} = k + cx + dy, \quad \dot{y} = l - x^2 - y^2, \quad (5.5)$$

for which

$$\begin{aligned} \mu_0 &= \mu_1 = 0, & \mu_2 &= (c^2 + d^2)(x^2 + y^2), \\ \mu_4 &= (x^2 + y^2) [(k^2 - c^2l)x^2 - 2cdlxy + (k^2 - d^2l)y^2]. \end{aligned}$$

Thus, according to the Remark 5.1, for  $\mu_2 \neq 0$  we obtain the case  $\Sigma_{13}$ .

Let us admit that condition  $\mu_2 = 0$  is satisfied. Then  $c = d = 0$  and for systems (5.5) we have  $\mu_3 = 0$ ,  $\mu_4 = k^2(x^2 + y^2)^2$ . This leads us to the case  $\Sigma_{30}$ .

## 5.3 Systems of type $(\mathbf{S}_{III})$

For canonical systems  $(\mathbf{S}_{III})$  one can calculate

$$\mu_0 = gh^2, \quad \kappa = -64h^2, \quad K = 2[g(g-1)x^2 + 2ghxy + h^2y^2].$$

It is quite clear that for  $\mu_0 \neq 0$  we have  $\Delta_S = 0$  and this leads us to the case  $\Sigma_3$ .

Suppose  $\mu_0 = 0$ . We examine the two cases:  $\kappa \neq 0$  and  $\kappa = 0$ .

### 5.3.1 Case $\kappa \neq 0$

Then  $h \neq 0$  which yields  $g = 0$  and thus for the systems  $(\mathbf{S}_{III})$  we have  $\gcd(p_2, q_2) = y$ . So, taking into consideration the Remark 5.1 and the fact that for the systems  $(\mathbf{S}_{III})$  the polynomial  $C_2(x, y) = x^2y$  we obtain the case  $\Sigma_7$  if  $\mu_1 \neq 0$ .

On the other hand the condition  $h \neq 0$  implies  $K \neq 0$ . Hence, by Proposition 4.1 and Lemma 5.1,  $\text{Supp } D_S(P, Q; Z)$  contains exactly one point  $[1 : 0 : 0]$  of the multiplicity  $(\Delta_S, 1)$ . Consequently we conclude that the quadratic system belongs to the set  $\Sigma_{14}$  (respectively,  $\Sigma_{21}$ ;  $\Sigma_{31}$ ) for  $\mu_1 = 0, \mu_2 \neq 0$  (respectively,  $\mu_{1,2} = 0, \mu_3 \neq 0$ ;  $\mu_{1,2,3} = 0, \mu_4 \neq 0$ ).

### 5.3.2 Case $\kappa = 0$

In this case  $h = 0$  and for systems  $(\mathbf{S}_{III})$  with  $p_2 = gx^2$ ,  $q_2 = (g-1)xy$  we have

$$\mu_0 = 0, \quad \mu_1 = dg(g-1)^2x, \quad L = 8gx^2,$$

and  $\gcd(p_2, q_2) = x$ . By Lemma 5.1 for  $\mu_1 \neq 0$  the quadratic systems belong to the set  $\Sigma_8$ .

Supposing  $\mu_1 = 0$  we shall consider two subcases:  $L \neq 0$  and  $L = 0$ .

**5.3.2.1 Subcase  $L \neq 0$ .** Then  $g \neq 0$  and hence  $\gcd(p_2, q_2) = x$  for  $g \neq 1$  and  $\gcd(p_2, q_2) = x^2$  for  $g = 1$ . Hence in both cases by Proposition 4.1 and Lemma 5.1,  $\text{Supp } D_S(P, Q; Z)$  contains exactly one point  $[0 : 1 : 0]$  whose multiplicity depends of the number of vanishing comitants  $\mu_i(a, x, y)$ . Therefore we conclude that the quadratic systems belong to the set  $\Sigma_{16}$  (respectively  $\Sigma_{24}$ ;  $\Sigma_{35}$ ) for  $\mu_2 \neq 0$  (respectively  $\mu_2 = 0, \mu_3 \neq 0$ ;  $\mu_{2,3} = 0, \mu_4 \neq 0$ ).

**5.3.2.2 Subcase  $L = 0$ .** For the systems  $(\mathbf{S}_{III})$  we have  $g = 0$  and applying the translation of the phase plane (to obtain  $e = f = 0$ ) these systems can be brought to the form

$$\dot{x} = k + cx + dy, \quad \dot{y} = l - xy. \quad (5.6)$$

For the systems (5.6) we have  $\mu_0 = \mu_1 = 0$  and

$$\mu_2 = -cdxy, \quad \mu_3 = -kxy(cx - dy), \quad \kappa_1 = -32d, \quad \mu_4 = -xy [c^2lx^2 + (2cdl - k^2)xy + d^2ly^2].$$

So, if  $\mu_2 \neq 0$  by the Remark 5.1 and Lemma 5.1 the systems (5.6) belong to the class  $\Sigma_{15}$ .

Let us suppose that the condition  $\mu_2 = 0$  holds.

**5.3.2.2.1** If  $\kappa_1 \neq 0$  then  $d \neq 0$  which implies  $c = 0$ . Then  $\mu_3 = dkxy^2$  and taking into consideration the factorization of the comitant  $\mu_4$ , we obtain the case  $\Sigma_{22}$  for  $\mu_3 \neq 0$  and the case  $\Sigma_{32}$  for  $\mu_3 = 0, \mu_4 \neq 0$ .

**5.3.2.2.2** Let us suppose  $\kappa_1 = 0$ . Then  $d = 0$  and for the system (5.6) we obtain

$$\mu_3 = -ckx^2y, \quad \mu_4 = -x^2y(c^2lx - k^2y), \quad K_1 = -cx^2y.$$

Therefore, if  $\mu_3 \neq 0$  by Remark 5.1 and Lemma 5.1 the systems (5.6) belong to the class  $\Sigma_{23}$ . If  $\mu_3 = 0$  we obtain  $ck = 0$  and we need to distinguish two cases:  $K_1 \neq 0$  and  $K_1 = 0$ .

The condition  $K_1 \neq 0$  yields  $c \neq 0$  and, hence,  $k = 0$ . This leads us to the case  $\Sigma_{34}$ . If  $K_1 = 0$  then  $c = 0$  and we obtain the case  $\Sigma_{33}$ .

## 5.4 Systems of type $(\mathbf{S}_IV)$

Note that for systems of the type  $(\mathbf{S}_IV)$  we have  $D_S(C, Z) = 3q$ . So,  $\text{Supp } D_S(P, Q; Z)$  could contain only the point  $[0 : 1 : 0]$ . By Lemma 5.1 its multiplicity depends of the number of the vanishing comitants  $\mu_i$ . Therefore we obtain that the quadratic system belongs to the set  $\Sigma_4$  (respectively  $\Sigma_9$ ;  $\Sigma_{17}$ ;  $\Sigma_{25}$ ;  $\Sigma_{36}$ ) for  $\mu_0 \neq 0$  (respectively for  $\mu_0 = 0, \mu_1 \neq 0$ ;  $\mu_{0,1} = 0, \mu_2 \neq 0$ ;  $\mu_{0,1,2} = 0, \mu_3 \neq 0$ ;  $\mu_{0,1,2,3} = 0, \mu_4 \neq 0$ ).

As all cases are examined, Theorem 5.1 is proved. ■

## 6 Divisors encoding the topology of singularities at infinity

We now need to consider the topological types of the singularities at infinity of quadratic systems. For this we shall introduce a third divisor at infinity:

**Definition 6.1.** *We call index divisor on the real line at infinity of  $\mathbb{R}^2$ , associated to a real system  $(S)$  such that  $Z \nmid C$ , the expression  $\sum i(w)w$  where  $w$  is a singular point on the line at infinity  $Z = 0$  of the system  $(S)$  and  $i(w)$  is the topological index (cf. [14]) of  $w$ , i.e.  $i(w)$  is the topological index of one of the two opposite singular points  $w, w'$  of  $\mathcal{V}$  on  $S^2$ .*

**Remark 6.1.** *This is a well defined divisor which could be extended trivially to a divisor  $\sum j(w)w$ ,  $w \in \{Z = 0\}$  on the line at infinity  $Z = 0$  of  $\mathbb{C}^2$  by letting*

$$j(w) = \begin{cases} i(w) & \text{if } w \in \mathbb{P}^2(\mathbb{R}) \\ 0 & \text{if } w \in \mathbb{P}^2(\mathbb{C}) \setminus \mathbb{P}^2(\mathbb{R}), \end{cases}$$

where we identify  $\mathbb{P}^2(\mathbb{R})$  with its image via the inclusion  $\mathbb{P}^2(\mathbb{R}) \hookrightarrow \mathbb{P}^2(\mathbb{C})$  induced by  $\mathbb{R} \hookrightarrow \mathbb{C}$ .

**Notation 6.1.** *We denote by  $I(S)$  the above divisor on  $Z = 0$  in  $\mathbb{P}^2(\mathbb{C})$ , i.e.  $I(S) = \sum j(w)w$ .*

**Notation 6.2.** *We denote by  $N_{\mathbb{C}}(S)$  (respectively, by  $N_{\mathbb{R}}(S)$ ) the total number of distinct singular points, be they real or complex (respectively, real), on the line at infinity  $Z = 0$  of the complex (respectively, real) foliation with singularities associated to  $(S)$ .*

We need to see how the divisor  $I(S) = \sum j(w)w$  and the divisors  $D_S(P, Q; Z) = \sum I_w(P, Q)w$  and  $D_S(C, Z) = \sum I_w(C, Z)p$  constructed in Section 3 are combined. For this we shall fuse these three divisors on the complex line at infinity into just one but with the values in the abelian group  $\mathbb{Z}^3$ :

**Notation 6.3.** *Let us consider the following divisor with the value in  $\mathbb{Z}^3$  on  $Z = 0$ :*

$$\mathcal{D}_S = \sum_w (I_w(C, Z), I_w(P, Q), j(w)) w$$

where  $w$  belongs to the line  $Z = 0$  of the complex projective plane.

We cannot detect the multiplicities of the singularities at infinity of a system  $S(\lambda)$  for the parameter value  $\lambda$  from just the phase portrait of  $S(\lambda)$ . On the other hand  $\mathcal{D}_{S(\lambda)}$  has dynamic qualities since it gives us some information about what could happen to the phase

portraits in the neighbourhood of  $\lambda$ . For example if  $w \in \{Z = 0\}$  and if  $I_w(P, Q) = 2$  for  $S(\lambda_0)$ , then we know that in the neighbourhood of  $\lambda_0$  the phase portraits of  $S(\lambda)$  will have 2 finite points arising from  $w$  in the neighbourhood of  $w$ .

We denote by  $\mathcal{H}'$  and  $\mathcal{H}$  the following sets:

$$\mathcal{H}' = \{X^2 + Y^2 + Z^2 = 1 \mid Z \geq 0\}, \quad \mathcal{H} = \{X^2 + Y^2 + Z^2 = 1 \mid Z > 0\}.$$

For  $(S)$  in **QS** satisfying the hypothesis of Theorem 5.1 let  $\sigma(S)$  be the set of all  $n_\infty = 2N_R(S)$  real singular points at infinity considered on the equator  $S^1$  of the Poincaré sphere.

We consider the function  $n_{sect} : \sigma(S) \rightarrow \mathbb{N}$  where  $n_{sect}(w)$  is the number of distinct local sectors of the point  $w \in S^1$  on  $\mathcal{H}$ .

Let  $w \in \sigma(S)$  and let  $\rho(S) = (w_1, w_2, \dots, w_{n_\infty})$  be the ordered sequence of singularities of  $S$  on  $S^1$ , enumerated when  $S^1$  is described in the positive sense and such that  $w_1 = w$ .

Let  $O_S(w) = (n_{sect}(w_1), n_{sect}(w_2), \dots, n_{sect}(w_{n_\infty}))$ . Then we have:

$$O_S(w_i) = (n_{sect}(w_i), n_{sect}(w_{i+1}), \dots, n_{sect}(w_{n_\infty}), n_{sect}(w_1), \dots, n_{sect}(w_{i-1})).$$

**Notation 6.4.** We denote by  $O(S)$  anyone of the sequences  $O_S(w_i)$ .

**Notation 6.5.** We denote by  $\max(n_{sect})$  the maximum value of the function  $n_{sect}$ , by  $N_{\max}(n_{sect}) = \#\{w \in S^1 \mid n_{sect}(w) = \max(n_{sect})\}$  and by  $N_{hsect}(S)$  the total number of hyperbolic sectors in  $\mathcal{H}'$  of singularities at infinity of a system  $(S) \in \mathbf{QS}_{\text{ess}}$ .

**Definition 6.2.** Let  $h_1(w_1)$  and  $h_2(w_2)$  be two distinct hyperbolic sectors of singularities at infinity  $w_1, w_2$  of a system  $(S) \in \mathbf{QS}_{\text{ess}}$ . (i) We say that  $h_1(w_1)$  and  $h_2(w_2)$  are finitely adjacent if  $w_1 = w_2 = w$  and the two sectors  $h_1(w_1)$  and  $h_2(w_2)$  have a common border which is a separatrix of  $w$  in the finite plane.

(ii) We say that  $h_1(w_1)$  and  $h_2(w_2)$  are adjacent at infinity if  $w_1$  and  $w_2$  are opposite points of  $S^1$  and  $w_1$  (also  $w_2$ ) as a point of  $S^2$  has two hyperbolic sectors with a common border, part of the equator.

**Notation 6.6.** We shall use the following notation

$$N_{hsect}^{f\infty a} = (N_{hsect}^{f-a}, N_{hsect}^{\infty-a}),$$

where  $N_{hsect}^{f-a}$  (respectively  $N_{hsect}^{\infty-a}$ ) is the total number of finitely adjacent couples of hyperbolic sectors (respectively adjacent at infinity).



## 7 Classification of quadratic differential systems according to their behavior in the neighborhood of infinity

The study of the geometry of the systems yields a simpler set of algebraic invariants than those used in [16]. We refine here the invariants which appeared in [16] so as to reveal the geometry of the systems.

We now need to relate the geometrical invariants defined in the previous section to their algebraic counterparts, i.e. the comitants and algebraic invariants.

To do this we construct below the  $GL$ -comitants which we need, by using the following basic ones:

$$\begin{aligned} C_i &= yp_i(x, y) - xq_i(x, y), \quad i = 0, 1, 2, \\ D_i &= \frac{\partial}{\partial x} p_i(x, y) + \frac{\partial}{\partial y} q_i(x, y), \quad i = 1, 2, \quad J_1 = \text{Jacob}(C_0, D_2), \\ J_2 &= \text{Jacob}(C_0, C_2), \quad J_3 = \text{Discrim}(C_1), \quad J_4 = \text{Jacob}(C_1, D_2). \end{aligned}$$

Using the comitants (4.1) and (5.1) we constructed in Sections 4 and 5 we define the following new polynomials:

$$\begin{aligned} N &= K + H, \quad R = L + 8K, \quad \kappa_2 = -J_1, \quad \xi = M - 2K, \\ K_2 &= 4 \text{Jacob}(J_2, \xi) + 3 \text{Jacob}(C_1, \xi) D_1 - \xi(16J_1 + 3J_3 + 3D_1^2), \\ K_3 &= 2C_2^2(2J_1 - 3J_3) + C_2(3C_0K - 2C_1J_4) + 2K_1(C_1D_2 + 3K_1). \end{aligned} \tag{7.1}$$

All these polynomials are  $GL$ -comitants, being obtained from simpler  $GL$ -comitants.

In the statement of the next Theorem *Figure  $j$*  for  $j=1, \dots, 40$  will denote a phase portrait in the vicinity of infinity of a quadratic system in  $\mathbf{QS}_{\text{ess}}$ . The notation for the figures in [16] was FIG  $j$ ,  $j = 1, \dots, 40$ . The correspondence between the two notations is indicated in columns 6 and 7 in Table 3.

In our next Theorem we relate the geometry at infinity of quadratic systems with algebraic and geometric invariants.

**Theorem 7.1. [The classification theorem]** *We consider here the family  $\mathbf{QS}_{\text{ess}}$  of all systems  $(S)$  in  $\mathbf{QS}$  which are essentially quadratic, i.e.  $\gcd(P, Q) = 1$  and  $Z \nmid C$ .*

**A.** *The phase portraits in the vicinity of infinity of the class  $\mathbf{QS}_{\text{ess}}$  are classified topologically by the integer-valued affine invariant  $\mathcal{J} = (O, N_{\text{hsect}}, N_{\text{hsect}}^{f\infty a})$  which expresses geo-*

metrical properties of the systems, e.g. number of real singularities, number of their sectors and the way in which these numbers are concatenated, etc. The classification appears in Table 3 with the corresponding phase portraits in Table 5, where they are listed for each value of  $N_{\mathbb{R}}(S)$  in order of increasing topological complexity.

**B.** The geometrical properties in the neighbourhood of infinity of quadratic systems ( $S$ ) in  $\mathbf{QS}_{\text{ess}}$  are expressed in terms of algebraic invariants and comitants as indicated in Table 4, which contains the full information regarding multiplicities and indices of the singularities at infinity for all quadratic differential systems in  $\mathbf{QS}_{\text{ess}}$ . The conditions appearing in the last column of Table 4 are affinely invariant.

The proof is based on the Theorem 5.1 as well as on the invariant classification of quadratic systems at infinity given in [16], subject to some corrections as we shall indicate below.

We point out that the affinely invariant conditions occurring in part **B** of the theorem, greatly simplify the analogous conditions in [16].

**Remark 7.1 (Corrections to [16]).** In the statement of Theorem 2 (a), b) in [3] (see page 92)  $\Delta_m > 0$  must be replaced by  $\Delta_m < 0$  and conversely. Since this theorem was used in [16] we have to note that several expressions in the sequences of the invariant conditions given in [16] must be taken with opposite sign, more precisely:

- FIG. 4: the inequality  $FS_1 > 0$  must be replaced by  $FS_1 < 0$ ;
- FIG. 5: the inequality  $FS_1 < 0$  must be replaced by  $FS_1 > 0$ ;
- FIG. 6: the inequality  $GA < 0$  must be replaced by  $GA > 0$ ;
- FIG. 7: the inequality  $GA > 0$  must be replaced by  $GA < 0$ ;
- FIG. 37: the inequality  $S_3 < 0$  must be replaced by  $S_3 > 0$ ,  $FS_1 < 0$ ;
- FIG. 38: the inequalities  $S_3 > 0$ ,  $FS_1 < 0$  must be replaced by  $S_3 > 0$ ,  $FS_1 < 0$ .

Furthermore the saddle-node given in FIG. 29 of [16] is not correctly placed. The correct phase portrait is given here in Figure 15.

*Proof of the Theorem 5.1.* **A.** The phase portraits in the vicinity of infinity of  $\mathbf{QS}_{\text{ess}}$  where obtained in [16]. All calculations were done again for this article and as we indicated in Remark 7.1, all phase portraits obtained in [16] with exception of FIG. 29 turned out to be correct. Figure 29 in [16] needed to be modified at one of its singularities and we give the respective corrected figure in Table 5 (Figure 15).

**Table 3**

$N_{\mathbb{R}}(S)$	$\max(n_{sect})$	$N_{\max}(n_{sect})$	$O(S)$	$N_{hsect}$	# of Figures		$N_{hsect}^{f\infty a}$		
					New	Old			
3	1	6	(1,1,1,1,1,1)	0	1	2			
	2	1	(2,1,1,1,1,1)	2	2	4			
		2	(2,2,1,1,1,1)	4	3	7			
			(2,1,2,1,1,1)	4	4	6			
			(2,1,1,2,1,1)	4	5	1			
		3	(2,2,1,1,2,1)	6	6	5			
		4	(2,2,1,2,2,1)	8	7	3			
	2	1	4	(1,1,1,1)	2	8		22	
1					9	12			
0					10	18			
2		1	1	(2,1,1,1)	3	11	15	(2,0)	
					2	12	26		(0,2)
						13	16		
					1	14	23		
					2	(2,2,1,1)	3		
		(2,1,2,1)	5	16			13		
			4	17		20			
			2	18		8	(0,0)		
		19		21					
		3	(2,1,2,2)	4	20	10	(2,2)		
					21	25	(2,0)		
		4	(2,2,2,2)	6	22	9			
		3	1	(3,1,1,1)	4	23		11	
					3	24		28	
				(3,1,2,1)	4	25		24	
					(3,2,1,2)	5		26	14
2			(3,1,3,1)	6		27		19	
				2	28	27			
			(3,2,3,2)	6	29	17			

**Table 3** (*continued*)

$N_{\mathbb{R}}(S)$	$\max(n_{sect})$	$N_{\max}(n_{sect})$	$O(S)$	$N_{hsect}$	# of Figures		$N_{hsect}^{f\infty a}$
					New	Old	
1	1	2	(1,1)	0	30	30	
	2	1	(2,1)	2	31	32	
				1	32	34	
				0	33	38	
		2	(2,2)	4	34	31	
				2	35	40	(2,0)
					36	39	(0,2)
	3	1	(3,1)	2	37	33	(2,0)
					38	37	(0,0)
			(3,2)	3	39	36	
				2	(3,3)	4	

**Table 4**

Figures	Value of $\mathcal{D}_S$	Necessary and sufficient conditions
Fig. 1	$(1, 0, 1)p + (1, 0, 1)q + (1, 0, 1)r$	$\eta > 0, \mu_0 < 0, \kappa > 0$
	$(1, 2, 1)p + (1, 0, 1)q + (1, 0, 1)r$	$\eta > 0, \mu_{0,1} = 0, \mu_2 < 0, \kappa > 0$
Fig. 2	$(1, 1, 0)p + (1, 0, 1)q + (1, 0, 1)r$	$\eta > 0, \mu_0 = 0, \mu_1 \neq 0, \kappa > 0$
	$(1, 3, 1)p + (1, 0, 1)q + (1, 0, 1)r$	$\eta > 0, \mu_{0,1,2} = 0, \mu_3 \neq 0, \kappa > 0$
	$(1, 2, 1)p + (1, 1, 1)q + (1, 0, 1)r$	$\eta > 0, \mu_{0,1,2} = \kappa = 0, \mu_3 K_1 < 0$
Fig. 3	$(1, 1, 0)p + (1, 1, 0)q + (1, 0, 1)r$	$\eta > 0, \mu_{0,1} = \kappa = 0, \mu_2 L < 0$
	$(1, 3, 0)p + (1, 1, 0)q + (1, 0, 1)r$	$\eta > 0, \mu_{0,1,2,3} = \kappa = 0, \mu_4 L < 0, K_1 \neq 0$
Fig. 4	$(1, 1, 0)p + (1, 1, 0)q + (1, 0, 1)r$	$\eta > 0, \mu_{0,1} = \kappa = 0, \mu_2 L > 0$
	$(1, 3, 0)p + (1, 1, 0)q + (1, 0, 1)r$	$\eta > 0, \mu_{0,1,2,3} = \kappa = 0, \mu_4 L > 0, K_1 \neq 0$
Fig. 5	$(1, 0, 1)p + (1, 0, 1)q + (1, 0, -1)r$	$\eta > 0, \mu_0 > 0$
	$(1, 2, 1)p + (1, 0, 1)q + (1, 0, -1)r$	$\eta > 0, \mu_{0,1} = 0, \mu_2 > 0, \kappa < 0$
	$(1, 4, 1)p + (1, 0, 1)q + (1, 0, -1)r$	$\eta > 0, \mu_{0,1,2,3} = 0, \mu_4 \neq 0, \kappa < 0$
	$(1, 0, 1)p + (1, 0, 1)q + (1, 2, -1)r$	$\eta > 0, \mu_{0,1} = 0, \mu_2 > 0, \kappa > 0$
	$(1, 0, 1)p + (1, 0, 1)q + (1, 4, -1)r$	$\eta > 0, \mu_{0,1,2,3} = 0, \mu_4 \neq 0, \kappa > 0$
	$(1, 2, 1)p + (1, 0, 1)q + (1, 2, -1)r$	$\eta > 0, \mu_{0,1,2,3} = 0, \mu_4 \neq 0, \kappa = K_1 = 0$

**Table 4** (*continued*)

Figures	Value of $\mathcal{D}_S$	Necessary and sufficient conditions
Fig. 6	$(1, 1, 0)p + (1, 0, 1)q + (1, 0, -1)r$	$\eta > 0, \mu_0 = 0, \mu_1 \neq 0, \kappa < 0$
	$(1, 3, 1)p + (1, 0, 1)q + (1, 0, -1)r$	$\eta > 0, \mu_{0,1,2} = 0, \mu_3 \neq 0, \kappa < 0$
	$(1, 2, 1)p + (1, 1, 1)q + (1, 0, -1)r$	$\eta > 0, \mu_{0,1,2} = \kappa = 0, \mu_3 K_1 > 0$
Fig. 7	$(1, 0, -1)p + (1, 0, 1)q + (1, 0, -1)r$	$\eta > 0, \mu_0 < 0, \kappa < 0$
	$(1, 2, -1)p + (1, 0, 1)q + (1, 0, -1)r$	$\eta > 0, \mu_{0,1} = 0, \mu_2 < 0, \kappa < 0$
Fig. 8	$(2, 2, 0)p + (1, 0, 1)q$	$\eta = 0, M \neq 0, \mu_{0,1} = \kappa = \kappa_1 = 0,$ $\mu_2 > 0, L > 0, K_2 < 0$
	$(2, 4, 0)p + (1, 0, 1)q$	$\eta = 0, M \neq 0, \mu_{0,1,2,3} = \kappa = \kappa_1 = 0,$ $\mu_4 > 0, L > 0, K = 0, K_2 < 0$
	$(2, 2, 0)p + (1, 2, 1)q$	$\eta = 0, M \neq 0, \mu_{0,1,2,3} = \kappa = \kappa_1 = 0$ $\mu_4 \neq 0, L = K_1 = 0, \kappa_2 < 0$
Fig. 9	$(2, 1, 1)p + (1, 0, 1)q$	$\eta = 0, M\mu_1 \neq 0, \mu_0 = \kappa = 0, L > 0, K < 0$
	$(2, 3, 1)p + (1, 0, 1)q$	$\eta = 0, M\kappa_1 L \neq 0, \mu_{0,1,2} = \kappa = 0, \mu_3 K_1 < 0$
	$(2, 1, 1)p + (1, 2, 1)q$	$\eta = 0, M \neq 0, \mu_{0,1,2} = \kappa = L = 0,$ $\kappa_1 \neq 0, \mu_3 K_1 < 0$
Fig. 10	$(2, 2, 2)p + (1, 0, 1)q$	$\eta = 0, M \neq 0, \mu_{0,1} = \kappa = \kappa_1 = 0,$ $\mu_2 < 0, L > 0, K < 0$
Fig. 11	$(2, 1, 1)p + (1, 1, 0)q$	$\eta = 0, M \neq 0, \mu_{0,1} = \kappa = L = 0, \mu_2 \neq 0$
	$(2, 1, 1)p + (1, 3, 0)q$	$\eta = 0, M \neq 0, \mu_{0,1,2,3} = \kappa = L = 0, \mu_4 \kappa_1 \neq 0$
Fig. 12	$(2, 2, 2)p + (1, 1, 0)q$	$\eta = 0, M \neq 0, \mu_{0,1,2} = \kappa = \kappa_1 = 0,$ $L = 0, \mu_3 K_1 < 0$
Fig. 13	$(2, 2, 1)p + (1, 0, 1)q$	$\eta = 0, M \neq 0, \mu_{0,1} = \kappa = 0, \mu_2 \neq 0, \kappa_1 L \neq 0$
	$(2, 4, 1)p + (1, 3, 0)q$	$\eta = 0, M\mu_4 \neq 0, \mu_{0,1,2,3} = \kappa = 0, \kappa_1 L \neq 0$
Fig. 14	$(2, 3, 1)p + (1, 0, 1)q$	$\eta = 0, M \neq 0, \mu_{0,1,2} = \kappa = \kappa_1 = 0,$ $\mu_3 \neq 0, L > 0, K < 0$
Fig. 15	$(2, 3, 1)p + (1, 1, 0)q$	$\eta = 0, M\mu_4 K_1 \neq 0, \mu_{0,1,2,3} = \kappa = \kappa_1 = L = 0$
Fig. 16	$(2, 1, 1)p + (1, 0, -1)q$	$\eta = 0, M\mu_1 \neq 0, \mu_0 = \kappa = 0, L < 0, N \leq 0$
	$(2, 1, 1)p + (1, 2, -1)q$	$\eta = 0, M \neq 0, \mu_{0,1,2} = \kappa = L = 0,$ $\kappa_1 \neq 0, \mu_3 K_1 > 0$

**Table 4** (*continued*)

Figures	Value of $\mathcal{D}_S$	Necessary and sufficient conditions
Fig. 17	$(2, 2, 2) p + (1, 0, -1) q$	$\eta = 0, M \neq 0, \mu_{0,1} = \kappa = \kappa_1 = 0, \mu_2 > 0, L < 0$
	$(2, 4, 2) p + (1, 0, -1) q$	$\eta = 0, M \neq 0, \mu_{0,1,2,3} = \kappa = \kappa_1 = 0, \mu_4 > 0, L < 0$
	$(2, 2, 2) p + (1, 2, -1) q$	$\eta = 0, M \neq 0, \mu_{0,1,2,3} = \kappa = \kappa_1 = L = K_1 = 0, \mu_4 \neq 0, \kappa_2 > 0$
Fig. 18	$(2, 0, 0) p + (1, 0, 1) q$	$\eta = 0, M \neq 0, \mu_0 > 0$
	$(2, 0, 0) p + (1, 2, 1) q$	$\eta = 0, M \neq 0, \mu_{0,1} = 0, \mu_2 > 0, \kappa \neq 0$
	$(2, 0, 0) p + (1, 4, 1) q$	$\eta = 0, M \neq 0, \mu_{0,1,2,3} = 0, \mu_4 \neq 0, \kappa \neq 0$
	$(2, 4, 0) p + (1, 0, 1) q$	$\eta = 0, M \neq 0, \mu_{0,1,2,3} = \kappa = \kappa_1 = 0,$ $\mu_4 > 0, L > 0, K \neq 0, R \geq 0$
		$\eta = 0, M \neq 0, \mu_{0,1,2,3} = \kappa = \kappa_1 = 0,$ $\mu_4 > 0, L > 0, K = 0, K_2 \geq 0$
Fig. 19	$(2, 2, 0) p + (1, 0, 1) q$	$\eta = 0, M \neq 0, \mu_{0,1} = \kappa = \kappa_1 = 0, \mu_2 > 0, L > 0, K_2 \geq 0$
Fig. 20	$(2, 0, 0) p + (1, 1, 0) q$	$\eta = 0, M \neq 0, \mu_0 = 0, \mu_1 \neq 0, \kappa \neq 0$
	$(2, 0, 0) p + (1, 3, 0) q$	$\eta = 0, M \neq 0, \mu_{0,1,2} = 0, \mu_3 \neq 0, \kappa \neq 0$
Fig. 21	$(2, 2, 0) p + (1, 1, 0) q$	$\eta = 0, M \neq 0, \mu_{0,1,2} = \kappa = \kappa_1 = L = 0, \mu_3 K_1 > 0$
Fig. 22	$(2, 0, 0) p + (1, 0, 1) q$	$\eta = 0, M \neq 0, \mu_0 < 0$
	$(2, 0, 0) p + (1, 2, 1) q$	$\eta = 0, M \neq 0, \mu_{0,1} = 0, \mu_2 < 0, \kappa \neq 0$
Fig. 23	$(2, 1, -1) p + (1, 0, 1) q$	$\eta = 0, M \neq 0, \mu_0 = \kappa = 0, \mu_1 \neq 0, L > 0, K > 0$
	$(2, 3, -1) p + (1, 0, 1) q$	$\eta = 0, M \neq 0, \mu_{0,1,2} = \kappa = 0, \kappa_1 L \neq 0, \mu_3 K_1 > 0$
Fig. 24	$(2, 4, 0) p + (1, 0, 1) q$	$\eta = 0, ML \neq 0, \mu_{0,1,2,3} = \kappa = \kappa_1 = 0, \mu_4 < 0$
Fig. 25	$(2, 3, -1) p + (1, 0, 1) q$	$\eta = 0, M \neq 0, \mu_{0,1,2} = \kappa = \kappa_1 = 0,$ $\mu_3 \neq 0, L > 0, K > 0$
Fig. 26	$(2, 1, 1) p + (1, 0, -1) q$	$\eta = 0, M \neq 0, \mu_0 = \kappa = 0, \mu_1 \neq 0, L < 0, N > 0$
	$(2, 3, 1) p + (1, 0, -1) q$	$\eta = 0, M \neq 0, \mu_{0,1,2} = \kappa = \kappa_1 = 0, \mu_3 \neq 0, L < 0$
Fig. 27	$(2, 2, -2) p + (1, 0, 1) q$	$\eta = 0, M \neq 0, \mu_{0,1} = \kappa = \kappa_1 = 0,$ $\mu_2 < 0, L > 0, K > 0$
Fig. 28	$(2, 4, 0) p + (1, 0, 1) q$	$\eta = 0, M \neq 0, \mu_{0,1,2,3} = \kappa = \kappa_1 = 0,$ $\mu_4 > 0, L > 0, K \neq 0, R < 0$
Fig. 29	$(2, 2, 0) p + (1, 0, -1) q$	$\eta = 0, M \neq 0, \mu_{0,1} = \kappa = \kappa_1 = 0, \mu_2 < 0, L < 0$

**Table 4** (*continued*)

Figures	Value of $\mathcal{D}_S$	Necessary and sufficient conditions
Fig. 30	$(1, 0, 1)p + (1, 0, 0)q^c + (1, 0, 0)r^c$	$\eta < 0, \mu_0 > 0$
	$(1, 2, 1)p + (1, 0, 0)q^c + (1, 0, 0)r^c$	$\eta < 0, \mu_{0,1} = 0, \mu_2 > 0, \kappa \neq 0$
	$(1, 4, 1)p + (1, 0, 0)q^c + (1, 0, 0)r^c$	$\eta < 0, \mu_{0,1,2,3} = 0, \mu_4 \neq 0, \kappa \neq 0$
	$(1, 0, 1)p + (1, 1, 0)q^c + (1, 1, 0)r^c$	$\eta < 0, \mu_{0,1} = \kappa = 0, \mu_2 \neq 0$
	$(1, 0, 1)p + (1, 2, 0)q^c + (1, 2, 0)r^c$	$\eta < 0, \mu_{0,1,2,3} = \kappa = 0, \mu_4 \neq 0$
	$(3, 0, 1)p$	$M = 0, \mu_0 > 0$
	$(3, 2, 1)p$	$M = 0, \mu_{0,1} = 0, \mu_2 > 0, K \neq 0, K_2 < 0$
		$M = 0, \mu_{0,1} = 0, \mu_2 > 0, K = 0$
	$(3, 4, 1)p$	$M = 0, \mu_{0,1,2,3} = 0, \mu_4 > 0, K_3 \geq 0$
Fig. 31	$(1, 1, 0)p + (1, 0, 0)q^c + (1, 0, 0)r^c$	$\eta < 0, \mu_0 = 0, \mu_1 \neq 0$
	$(1, 3, 0)p + (1, 0, 0)q^c + (1, 0, 0)r^c$	$\eta < 0, \mu_{0,1,2} = 0, \mu_3 \neq 0$
	$(3, 3, 0)p$	$M = 0, \mu_{0,1,2} = K = 0, \mu_3 K_1 > 0, K_3 \geq 0$
Fig. 32	$(3, 2, 1)p$	$M = 0, \mu_{0,1} = 0, \mu_2 > 0, K \neq 0, K_2 \geq 0$
	$(3, 4, 1)p$	$M = 0, \mu_{0,1,2,3} = K = 0, \mu_4 > 0, K_3 < 0$
Fig. 33	$(3, 3, 2)p$	$M = 0, \mu_{0,1,2} = K = 0, \mu_3 K_1 < 0$
Fig. 34	$(1, 0, -1)p + (1, 0, 0)q^c + (1, 0, 0)r^c$	$\eta < 0, \mu_0 < 0$
	$(1, 2, -1)p + (1, 0, 0)q^c + (1, 0, 0)r^c$	$\eta < 0, \mu_{0,1} = 0, \mu_2 < 0, \kappa \neq 0$
	$(3, 0, -1)p$	$M = 0, \mu_0 < 0$
Fig. 35	$(3, 4, 1)p$	$M = 0, \mu_{0,1,2,3} = 0, \mu_4 < 0$
Fig. 36	$(3, 4, 1)p$	$M = 0, \mu_{0,1,2,3} = 0, \mu_4 > 0, K \neq 0, K_3 < 0$
Fig. 37	$(3, 1, 0)p$	$M = 0, \mu_0 = 0, \mu_1 \neq 0$
	$(3, 3, 0)p$	$M = 0, \mu_{0,1,2} = 0, \mu_3 K \neq 0, K_3 > 0$
Fig. 38	$(3, 3, 0)p$	$M = 0, \mu_{0,1,2} = K = 0, \mu_3 K_1 > 0, K_3 < 0$
Fig. 39	$(3, 3, 0)p$	$M = 0, \mu_{0,1,2} = 0, \mu_3 K \neq 0, K_3 < 0$
Fig. 40	$(3, 2, -1)p$	$M = 0, \mu_{0,1} = 0, \mu_2 < 0$

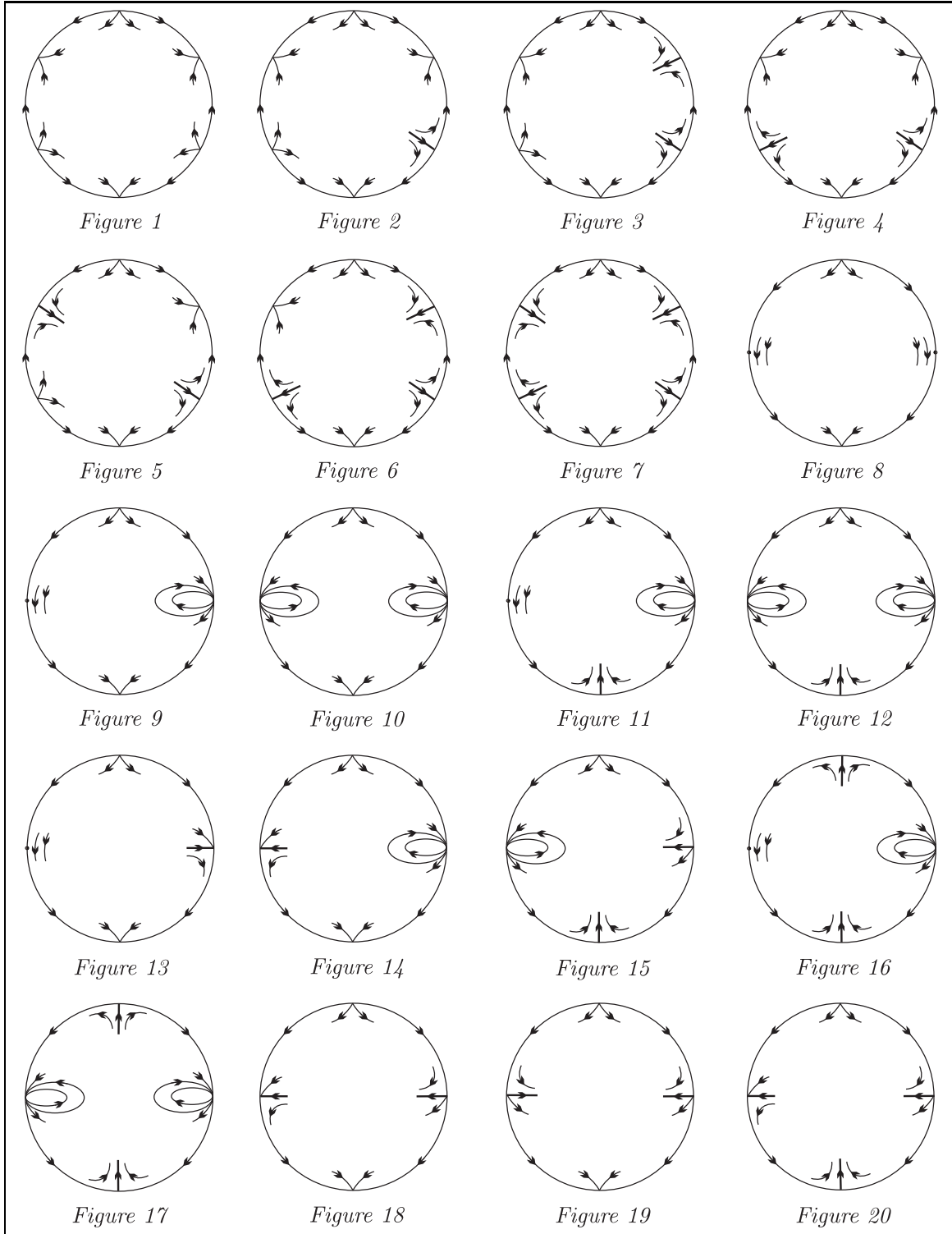


Table 5



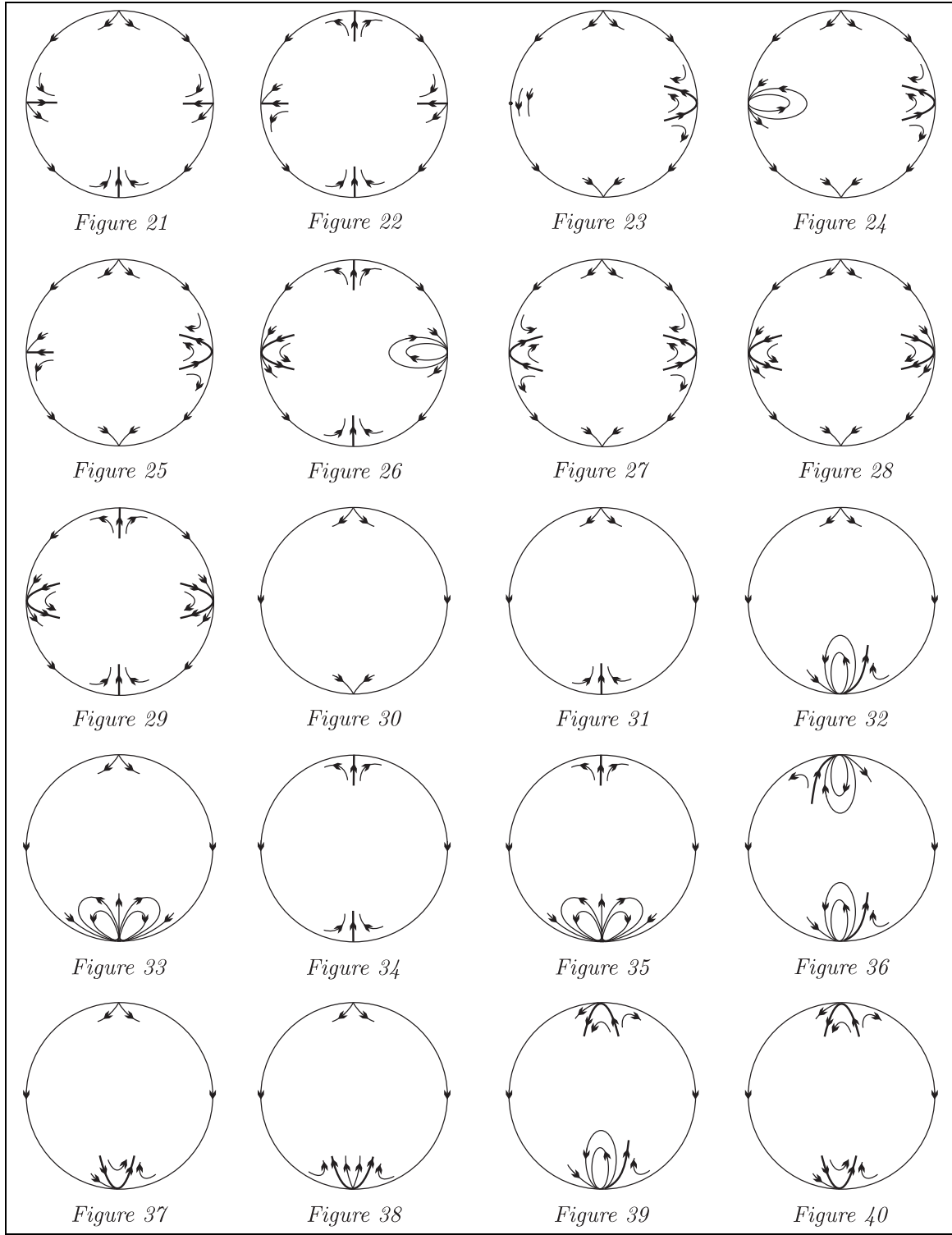


Table 5 (*continued*)

In [16] (see pages 481–484), the phase portraits appeared as they were obtained from calculations and not listed according to their geometry. To draw attention to the geometry we list them here for each possible value of  $N_{\mathbb{R}}(S)$  according to their topological complexity. In Table 3 we first place the number  $N_{\mathbb{R}}(S)$  of real singularities of the real foliation on  $\mathbb{P}^2(\mathbb{R})$ , followed by the maximum number  $\max(n_{sect})$  of sectors of singularities. Although these numbers could be read on the value of  $O(S)$ , we place them in separate columns as they are important invariants for the geometry at infinity of the systems. We complete the table going through all phase portraits and listing  $O(S)$  which by itself determines uniquely 27 of the 40 phase portraits. To distinguish the remaining 13 phase portraits we use the invariant  $N_{hsect}^{f\infty a} = (N_{hsect}^{f-a}, N_{hsect}^{\infty-a})$  whose values we place in the last column, thus completing the classification.

**B.** As in the proof of part **A** we use the results in [16] subject to the modifications in Remark 7.1. Since some letters appear both here and in [16] but not always with the same meaning, we shall use the convention to apply "tilde" to letters which are used to denote comitants in [16].

The proof of part **B** proceeds in 3 steps:

I) In this step we replace the conditions in [16] subject to the modifications in Remark 7.1 with conditions involving newly defined comitants and invariants as we shall indicate below.

II) In this part we simplify the conditions obtained in step I) in order to obtain the corresponding conditions in the last column of Table 4.

III) We prove that these last conditions are affinely invariant.

*Proof of step I.* First of all we shall prove that the comitants used in [16] (see Appendix) can be replaced respectively by the comitants used here as follows:

$$\begin{aligned} \tilde{\mu} &\Rightarrow \mu_0; & \tilde{H} &\Rightarrow \mu_1; & \tilde{G} &\Rightarrow \mu_2; & \tilde{F} &\Rightarrow \mu_3; & \tilde{V} &\Rightarrow \mu_4; & \tilde{L} &\Rightarrow C_2; & \tilde{M} &\Rightarrow M; \\ \tilde{\eta} &\Rightarrow \eta; & \tilde{\theta} &\Rightarrow \kappa; & \tilde{N} &\Rightarrow K; & \tilde{S}_1 &\Rightarrow K_1; & \tilde{A} &\Rightarrow L; & \tilde{A} + 4\tilde{N} &\Rightarrow R; \\ \tilde{A} + \tilde{N} &\Rightarrow N; & \tilde{\sigma} &\Rightarrow \kappa_1; & \tilde{S}_2 &\Rightarrow K_2; & \tilde{S}_3 &\Rightarrow K_3; & \tilde{S}_4 &\Rightarrow \kappa_2; \end{aligned} \quad (7.2)$$

Indeed, firstly the following relations among the comitants (7.2) hold:

$$\begin{aligned} \mu_0 &= \tilde{\mu}, & \mu_1 &= 2\tilde{H}, & \mu_2 &= \tilde{G}, & \mu_3 &= \tilde{F}, & \mu_4 &= \tilde{V}, & C_2 &= \tilde{L}, & M &= 8\tilde{M}, \\ \eta &= \tilde{\eta}, & \kappa &= 64\tilde{\theta}, & K &= 4\tilde{N}, & K_1 &= \tilde{S}_1, & L &= 8\tilde{A}, & R &= 8(\tilde{A} + 4\tilde{N}). \end{aligned} \quad (7.3)$$

Therefore we only have to compare the conditions involving the comitants

$$N, \quad \kappa_1, \quad \kappa_2, \quad K_2, \quad K_3 \quad (7.4)$$

and show the corresponding equivalence with the conditions involving the comitants

$$\tilde{A} + \tilde{N}, \quad \tilde{\sigma}, \quad \tilde{S}_4, \quad \tilde{S}_2, \quad \tilde{S}_3 \quad (7.5)$$

in [16], respectively.

We point out that all comitants (7.5) are only used for the systems  $(\mathbf{S}_{III})$  and  $(\mathbf{S}_{IV})$ . So, in what follows we shall examine each one of this cases.

We first consider the systems of the form  $(\mathbf{S}_{III})$ .

In this case we have four singularities on the equator (i.e.  $\eta = 0$ ,  $M \neq 0$ ). The phase portraits in the vicinity of infinity of these systems are given by one of the Figures 8-29 both here and in [16]. One can observe, that all comitants (7.4) (respectively, (7.5)) are used for systems  $(\mathbf{S}_{III})$  only in the case when  $\kappa = 0$  (respectively,  $\tilde{\theta} = 0$ ). In this case for systems  $(\mathbf{S}_{III})$  the condition  $\kappa = -64h^2 = 0$  yields  $h = 0$  and we obtain the systems

$$\dot{x} = k + cx + dy + gx^2, \quad \dot{y} = l + ex + fy + (g - 1)xy, \quad (7.6)$$

for which  $L = 8gx^2$  and

$$\kappa_1 = -32d, \quad \tilde{\sigma} = -\frac{d}{4}(5g^2 - 2g + 1); \quad N = (g - 1)(g + 1)x^2, \quad \tilde{A} + \tilde{N} = \frac{1}{2}g(g + 1)x^2.$$

Clearly, the condition  $\kappa_1 = 0$  is equivalent to  $\tilde{\sigma} = 0$ . We now compare the signs of  $N$  and  $\tilde{A} + \tilde{N}$ . As in Table 4 the comitant  $N$  appears only in two cases (i.e. Figures 16 and 26) and in these cases the condition  $L < 0$  (i.e.  $g < 0$ ) is used, from the expressions of  $N$  and  $\tilde{A} + \tilde{N}$  above we obtain  $\text{sign}(N) = \text{sign}(\tilde{A} + \tilde{N})$ .

We observe from Table 4 that the comitant  $K_2$  is applied for systems  $(\mathbf{S}_{III})$  only when  $\kappa = \kappa_1 = 0$ ,  $L \neq 0$ . Since  $\kappa_1 = 0$  implies  $d = 0$  the systems (7.6) become

$$\dot{x} = k + cx + dy + gx^2, \quad \dot{y} = l + ex + fy + (g - 1)xy, \quad (7.7)$$

and we calculate  $K_2 = 48(g^2 - g + 2)(c^2 - 4gk)x^2$ ,  $\tilde{S}_2 = 2g^2(c^2 - 4gk)x^2$ . Hence,  $K_2$  has a well determined sign and since for every  $g$  we have  $g^2 - g + 2 > 0$ , from  $L \neq 0$  we obtain  $\text{sign}(K_2) = \text{sign}(\tilde{S}_2)$ .

We note that the invariant  $\kappa_2(a)$  is here used only to distinguish Figures 8 and 17 in the case when systems  $(\mathbf{S}_{III})$  belong to the class  $\Sigma_{33}$  in Table 2. Since for this class the conditions  $\kappa = L = K_1 = 0$  hold for systems  $(\mathbf{S}_{III})$ , we obtain respectively  $h = g = c^2 + d^2 = 0$ . So, the systems  $(\mathbf{S}_{III})$  become

$$\dot{x} = k, \quad \dot{y} = l + ex + fy - xy, \quad (7.8)$$

for which  $\kappa_2 = -k$ ,  $\tilde{S}_4 = -2k$  and, hence,  $\text{sign}(\kappa_2) = \text{sign}(\tilde{S}_4)$ .

It remains to consider systems of the form  $(\mathbf{S}_N)$ . For the Figures 30-40 which can occur for this class of systems, only the comitants  $\tilde{S}_2$  and  $\tilde{S}_3$  of (7.5) were used in [16]. Hence we only have to examine the conditions given in terms of comitants  $K_2$  and  $K_3$  from (7.4).

We observe that the comitant  $K_2$  is used to distinguish Figures 30 and 32 when we also have  $K \neq 0$ . In this case the systems  $(\mathbf{S}_N)$  belong to the class  $\Sigma_{17}$  in Table 2 with conditions  $\mu_0 = \mu_1 = 0$ . For systems  $(\mathbf{S}_N)$  we have  $\mu_0 = -8h^3$ . Hence  $h = 0$  and the systems  $(\mathbf{S}_N)$  become

$$\dot{x} = k + cx + dy + 2gx^2, \quad \dot{y} = l + ex + fy - x^2 + 2gxy, \quad (7.9)$$

for which  $K = 2g^2x^2$ ,  $\mu_1 = 8dg^3x$ . As  $K \neq 0$ , the condition  $\mu_1 = 0$  implies  $d = 0$  and we obtain the systems

$$\dot{x} = k + cx + dy + 2gx^2, \quad \dot{y} = l + ex + fy - x^2 + 2gxy, \quad (7.10)$$

for which we have:  $K_2 = 24g^2(c^2 - 8gk)x^2$ ,  $\tilde{S}_2 = 4g^2(c^2 - 8gk)x^2$ . Thus, in the case under consideration the comitant  $K_2$  has a well determined sign and  $\text{sign}(K_2) = \text{sign}(\tilde{S}_2)$ .

We examine now the comitant  $K_3$  which is applied for systems  $(\mathbf{S}_N)$  only in the cases when  $\Delta_S \geq 3$ , i.e.  $\mu_{0,1,2} = 0$ . So, we shall consider the systems (7.9) for which  $\mu_0 = 0$  and we examine two subcases:  $K \neq 0$  and  $K = 0$ .

If  $K \neq 0$  then  $g \neq 0$  and for the systems (7.9) the condition  $\mu_1 = 0$  gives  $d = 0$ . Moreover we may assume  $e = f = 0$  via a translation. So, we obtain the systems

$$\dot{x} = k + cx + 2gx^2, \quad \dot{y} = l - x^2 + 2gxy, \quad (7.11)$$

for which  $\mu_2 = 8g^3kx^2$  and as  $g \neq 0$  the condition  $\mu_2 = 0$  yields  $k = 0$ . Then for the systems (7.11) we obtain  $K_3 = -12g^2lx^6$ ,  $\tilde{S}_3 = -12g^2lx^6$ . Hence  $K_3$  has a well determined sign and  $\text{sign}(K_3) = \text{sign}(\tilde{S}_3)$ .

Assume now  $K = 0$ , i.e.  $g = 0$  and for the systems (7.9) we obtain  $\mu_1 = 0$ ,  $\mu_2 = d^2x^2$ . Thus, the condition  $\mu_2 = 0$  yields  $d = 0$  and we obtain the following systems

$$\dot{x} = k + cx, \quad \dot{y} = l + ex + fy - x^2, \quad (7.12)$$

for which  $K_3 = 3f(2c - f)x^6 = \tilde{S}_3$ . ■

*Proof of step II.* We show below how some of the conditions in [16] can be substituted by simpler ones in Table 4. To do this we shall prove the following five lemmas.

**Lemma 7.1.** *Let  $\tilde{\mathfrak{C}}$  be the conjunction of the all the conditions:  $\tilde{\eta} = \tilde{\mu} = \tilde{H} = \tilde{\theta} = \tilde{\sigma} = 0$  and  $\tilde{M}\tilde{G}\tilde{A} \neq 0$ . Let  $\mathfrak{C}$  be the conjunction of the following conditions:  $\eta = \mu_0 = \mu_1 = \kappa = \kappa_1 = 0$  and  $M\mu_2L \neq 0$ . We have the following equivalences:*

$$\begin{aligned}
\text{Figure 8 : } \tilde{\mathfrak{C}}, \tilde{G} \neq 0, \tilde{A} > 0, \tilde{S}_2 < 0 & \Leftrightarrow \mathfrak{C}, \mu_2 > 0, L > 0, K_2 < 0; \\
\text{Figure 10 : } \tilde{\mathfrak{C}}, \tilde{G} < 0, \tilde{A} > 0, \tilde{S}_2 > 0, \tilde{N} < 0 & \Leftrightarrow \mathfrak{C}, \mu_2 < 0, L > 0, K < 0; \\
\text{Figure 17 : } \tilde{\mathfrak{C}}, \tilde{G} \neq 0, \tilde{A} < 0, (\tilde{S}_2 \leq 0) \vee (\tilde{G} > 0, \tilde{S}_2 > 0) & \Leftrightarrow \mathfrak{C}, \mu_2 > 0, L < 0; \\
\text{Figure 19 : } \tilde{\mathfrak{C}}, \tilde{G} \neq 0, \tilde{A} > 0, (\tilde{S}_2 = 0) \vee (\tilde{G} > 0, \tilde{S}_2 > 0) & \Leftrightarrow \mathfrak{C}, \mu_2 > 0, L > 0, K_2 \geq 0; \\
\text{Figure 27 : } \tilde{\mathfrak{C}}, \tilde{G} < 0, \tilde{A} > 0, \tilde{S}_2 > 0, \tilde{N} > 0 & \Leftrightarrow \mathfrak{C}, \mu_2 < 0, L > 0, K > 0; \\
\text{Figure 29 : } \tilde{\mathfrak{C}}, \tilde{G} < 0, \tilde{A} < 0, \tilde{S}_2 > 0 & \Leftrightarrow \mathfrak{C}, \mu_2 < 0, L < 0.
\end{aligned}$$

*Proof:* According to (7.3) the conditions  $\tilde{\mathfrak{C}}$  and  $\mathfrak{C}$  are equivalent. We are in the class of the systems  $(\mathbf{S}_{III})$  for which we must apply the conditions on the right, i.e.  $\mu_0 = \mu_1 = 0$ ,  $\mu_2 \neq 0$ , and  $\kappa = \kappa_1 = 0$ ,  $L \neq 0$ . For the systems  $(\mathbf{S}_{III})$  we have  $\kappa = -64h^2$ ,  $\kappa_1 = -32d$  and hence conditions  $\kappa = \kappa_1 = 0$  yield  $h = d = 0$ . Then

$$\mu_0 = \mu_1 = 0, \quad \mu_2 = g[f^2g + cf(g-1) + k(g-1)^2]x^2 \neq 0$$

and since  $g \neq 0$  we may assume  $c = 0$  via a translation. Hence we get the systems

$$\dot{x} = k + gx^2, \quad \dot{y} = l + ex + fy + (g-1)xy, \quad (7.13)$$

for which

$$\begin{aligned}
\mu_{0,1} = 0, \quad \mu_2 = g[f^2g + k(g-1)^2]x^2\tilde{G} \neq 0, \quad L = gx^2 = 8\tilde{A} \neq 0, \\
K = 2g(g-1)x^2 = 4\tilde{N}, \quad K_2 = -192gk(g^2 - g + 2)x^2, \quad \tilde{S}_2 = -8g^3k.
\end{aligned} \quad (7.14)$$

We observe, that  $\text{sign}(K_2) = \text{sign}(\tilde{S}_2)$  because the discriminant of the quadratic polynomial  $g^2 - g + 2$  is negative. We shall consider two cases:  $L < 0$  and  $L > 0$ .

**Case  $L < 0$ .** If  $\mu_2 < 0$  (then  $\tilde{G} < 0$ ) from (7.14) it follows that  $\tilde{S}_2 > 0$  and hence we obtain the conditions indicates on the left in the lemma, which correspond to Figure 29. Thus the conditions  $L < 0$  and  $\mu_2 < 0$  lead to Figure 29.

Assume  $\mu_2 > 0$  (then  $\tilde{G} > 0$ ). If either  $K_2 > 0$  (then  $\tilde{S}_2 > 0$ ) or  $K_2 \leq 0$  (then  $\tilde{S}_2 \leq 0$ ) we obtain the conditions on the left for Figure 17. Taking into account that for  $\mu_2 \neq 0$  from (7.14) it follows that the condition  $\tilde{S}_2 \leq 0$  implies  $\mu_2 > 0$  (then  $\tilde{G} > 0$ ) we conclude, that the conditions  $L < 0$  and  $\mu_2 > 0$  lead to Figure 17.

**Case  $L > 0$ .** Suppose firstly  $\mu_2 < 0$ . Then  $\tilde{G} < 0$  and from (7.14) we have  $\tilde{S}_2 > 0$  and  $\tilde{N} \neq 0$  (i.e.  $K \neq 0$ ). Hence we obtain the conditions for Figure 10 (on the left in the lemma) if  $K < 0$  and for Figure 27 if  $K > 0$ .

Assume now  $\mu_2 > 0$  (then  $\tilde{G} > 0$ ). From (7.14) we obtain  $\tilde{S}_2 \leq 0$  (then  $K_2 \leq 0$ ) which yields  $\mu_2 > 0$ . Hence we conclude, that the conditions  $L > 0$ ,  $\mu_2 > 0$ ,  $K_2 \geq 0$  lead to the Figure 19, whereas the conditions  $L > 0$ ,  $\mu_2 > 0$ ,  $K_2 < 0$  lead to the Figure 8. ■

**Lemma 7.2.** *Let  $\tilde{\mathfrak{C}}_1$  be the conjunction of the following conditions:  $\tilde{\eta} = \tilde{\mu} = \tilde{H} = \tilde{G} = \tilde{F} = \tilde{\theta} = \tilde{\sigma} = 0$  and  $\tilde{M}\tilde{V}\tilde{A} \neq 0$ . Let  $\mathfrak{C}_1$  be the conjunction of the following conditions:  $\eta = \mu_0 = \mu_1 = \mu_2 = \mu_3 = \kappa = \kappa_1 = 0$  and  $M\mu_4L \neq 0$ . We have the following equivalences:*

$$\begin{aligned}
\text{Figure 8 : } & \tilde{\mathfrak{C}}_1, \tilde{V} \neq 0, \tilde{A} \neq 0, \tilde{N} = 0, \tilde{S}_2 < 0 & \Leftrightarrow & \mathfrak{C}_1, \mu_4 > 0, L > 0, K_2 < 0; \\
\text{Figure 17 : } & \tilde{\mathfrak{C}}_1, \tilde{V} \neq 0, \tilde{N} \neq 0, \tilde{A} < 0 & \Leftrightarrow & \mathfrak{C}_1, \mu_4 > 0, L < 0; \\
\text{Figure 18 : } & \left[ \begin{array}{l} \tilde{\mathfrak{C}}_1, \tilde{A}\tilde{V} \neq 0, (\tilde{N} = 0, \tilde{S}_2 = 0) \\ \vee (\tilde{N} \neq 0, \tilde{A} > 0, \tilde{A} + 4\tilde{N} \geq 0) \\ \vee (\tilde{N} = 0, \tilde{V} > 0, \tilde{S}_2 > 0) \end{array} \right] & \Leftrightarrow & \left[ \begin{array}{l} \mathfrak{C}_1, \mu_4 > 0, L > 0, \\ (R \geq 0, K \neq 0) \vee \\ (K_2 \geq 0, K = 0) \end{array} \right]; \\
\text{Figure 24 : } & \tilde{\mathfrak{C}}_1, \tilde{N} = 0, \tilde{A} \neq 0, \tilde{V} < 0 & \Leftrightarrow & \mathfrak{C}_1, \mu_4 < 0, L \neq 0; \\
\text{Figure 28 : } & \tilde{\mathfrak{C}}_1, \tilde{V}\tilde{N} \neq 0, \tilde{A} > 0, \tilde{A} + 4\tilde{N} < 0 & \Leftrightarrow & \mathfrak{C}_1, \mu_4 > 0, L > 0, R < 0.
\end{aligned}$$

*Proof:* We are in the class of systems  $(\mathbf{S}_{III})$  for which we must set the conditions  $\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0$ ,  $\mu_4 \neq 0$ , and  $\kappa = \kappa_1 = 0$ ,  $L = 8\tilde{A} \neq 0$ . It was shown before (see page 37) that for the systems  $(\mathbf{S}_{III})$  the conditions  $\kappa = \kappa_1 = 0$  yield  $h = d = 0$ . Then  $L = gx^2 \neq 0$  and  $K = 2g(g-1)x^2$  and we shall construct two canonical forms corresponding to the cases  $K \neq 0$  and  $K = 0$ .

Assume firstly  $K \neq 0$ . Then  $g-1 \neq 0$  and we may assume  $e = f = 0$  due to a translation. Therefore considering the conditions  $h = d = e = f = 0$ , for the systems  $(\mathbf{S}_{III})$  calculations yield:  $\mu_0 = \mu_1 = 0$ ,  $\mu_2 = gk(g-1)^2$  and by  $g(g-1) \neq 0$  the condition  $\mu_2 = 0$  yields  $k = 0$ . This implies  $\mu_3 = -clg(g-1)x^3$ ,  $\mu_4 = lx^3[lg^2x + c^2(g-1)y]$ . Hence, the conditions  $\mu_3 = 0$  and  $\mu_4 \neq 0$  yield  $c = 0$  and we get the systems

$$\dot{x} = gx^2, \quad \dot{y} = l + (g-1)xy, \quad (7.15)$$

for which

$$\begin{aligned}
\mu_{0,1,2,3} = 0, \mu_4 = g^2l^2x^4 = \tilde{V}, L = 8gx^2 = 8\tilde{A} \neq 0, K_2 = 0 = \tilde{S}_2, \\
K = 2g(g-1)x^2 = 4\tilde{N} \neq 0, R = 8g(2g-1)x^2 = 8(\tilde{A} + 4\tilde{N}).
\end{aligned} \quad (7.16)$$

Suppose now that the condition  $K = 2g(g-1)x^2 = 0$  holds. Since  $L = gx^2 \neq 0$  this yields  $g = 1$  and we may assume  $c = 0$  via a translation. Then we obtain  $\mu_2 = f^2x^2 = 0$  which implies  $f = 0$  and we get the systems

$$\dot{x} = k + x^2, \quad \dot{y} = l + ex, \quad (7.17)$$

for which

$$\begin{aligned} \mu_{0,1,2,3} = 0, \mu_4 = (l^2 + ke^2)x^4 = \tilde{V} \neq 0, L = 8x^2 = 8\tilde{A}, \\ K = 0 = \tilde{N}, R = 8x^2 = 8(\tilde{A} + 4\tilde{N}), K_2 = -384kx^2 = 48\tilde{S}_2. \end{aligned} \quad (7.18)$$

We shall consider two cases:  $\mu_4 < 0$  and  $\mu_4 > 0$ .

**Case  $\mu_4 < 0$ .** Then  $\tilde{V} < 0$  and from (7.16) and (7.18) we have the conditions  $\tilde{N} = 0$  and  $\tilde{S}_2 > 0$ . Hence the conditions  $\mu_4 < 0$  and  $L \neq 0$  lead to the conditions in the lemma corresponding to Figure 24.

**Case  $\mu_4 > 0$ .** In this case  $\tilde{V} > 0$  and we shall examine two subcases:  $L < 0$  and  $L > 0$ .

**Subcase  $L < 0$ .** Then  $\tilde{A} < 0$ . From (7.16) and (7.18) we conclude that  $\tilde{N} \neq 0$  and we obtain the conditions corresponding to Figure 17. Hence we conclude that for  $\mu_4 > 0$  and  $L < 0$  we get Figure 17.

**Subcase  $L > 0$ .** Hence  $\tilde{A} > 0$ .

a) If  $R < 0$  (then  $\tilde{A} + 4\tilde{N} < 0$ ) from (7.16) and (7.18) we obtain  $\tilde{N} \neq 0$  and hence we get the conditions for Figure 28.

b) Assume now  $R \geq 0$ . If  $K \neq 0$  (then  $\tilde{N} \neq 0$ ) we obtain one sequence of conditions for Figure 18, and namely:  $\tilde{N} \neq 0$ ,  $\tilde{A} > 0$  and  $\tilde{A} + 4\tilde{N} \geq 0$ .

Suppose  $K = 0$  (i.e.  $\tilde{N} = 0$ ). If in addition  $K_2 < 0$  (then  $\tilde{S}_2 < 0$ ) then we obtain the conditions for Figure 8. From (7.16) and (7.18) we obtain that the condition  $K_2 < 0$  implies  $\tilde{N} = 0$ . Then we conclude, that for  $\mu > 0$ ,  $L > 0$  and  $K_2 < 0$  we obtain the conditions for Figure 8.

Assuming  $K_2 \geq 0$  (then  $\tilde{S}_2 \geq 0$ ) and taking into account that we are in the case  $\mu_4 > 0$ , we get two of the series of conditions for Figure 18, which can be combined into the following series:  $\mu_4 > 0$ ,  $K = 0$ ,  $L > 0$ ,  $K_2 \geq 0$ . ■

**Lemma 7.3.** *Let  $\tilde{\mathfrak{C}}_2$  be the conjunction of all the conditions:  $\tilde{M} = \tilde{\mu} = \tilde{H} = 0$  and  $\tilde{L}\tilde{G} \neq 0$ . Let  $\mathfrak{C}_2$  be the conjunction of the following conditions:  $M = \mu_0 = \mu_1 = 0$  and  $C_2\mu_2 \neq 0$ . We have the following equivalences:*

$$\text{Figure 30 : } \tilde{\mathfrak{C}}_2, \tilde{G} \neq 0, (\tilde{N} \neq 0, \tilde{S}_2 < 0) \vee (\tilde{N} = 0) \Leftrightarrow \mathfrak{C}_2, \mu_2 > 0, (K \neq 0, K_2 < 0) \vee (K = 0);$$

$$\text{Figure 32 : } \tilde{\mathfrak{C}}_2, \tilde{G}\tilde{N} \neq 0, (\tilde{G} > 0, \tilde{S}_2 > 0) \vee (\tilde{S}_2 = 0) \Leftrightarrow \mathfrak{C}_2, \mu_2 > 0, K \neq 0, K_2 \geq 0;$$

$$\text{Figure 40 : } \tilde{\mathfrak{C}}_2, \tilde{G} < 0, \tilde{N} \neq 0, \tilde{S}_2 > 0 \Leftrightarrow \mathfrak{C}_2, \mu_2 < 0.$$

*Proof:* We are in the class of systems  $(\mathbf{S}_N)$  for which we must set the conditions  $\mu_0 = \mu_1 = 0$ ,

$\mu_2 \neq 0$ . We have  $\mu_0 = -h^3 = 0$  which implies  $h = 0$  and then  $\mu_1 = dg^3x$  and  $K = 2gx^2$ . We shall consider two subcases:  $K \neq 0$  and  $K = 0$ .

Assume firstly  $K \neq 0$ . Then  $g \neq 0$  and the condition  $\mu_1 = 0$  yields  $d = 0$ . We can assume  $g = 1$  and  $e = f = 0$  due to the rescaling  $x \rightarrow x/g$ ,  $y \rightarrow y/g^2$  and a translation. Then we get the systems

$$\dot{x} = k + cx + x^2, \quad \dot{y} = l - x^2 + xy, \quad (7.19)$$

for which

$$\mu_{0,1} = 0, \quad \mu_2 = kx^2 = \tilde{G} \neq 0, \quad K = 2x^2 = 4\tilde{N}, \quad K_2 = 48(c^2 - 4k)x^2 = 24\tilde{S}_2. \quad (7.20)$$

Admit now  $K = 0$ . Hence  $g = 0$  and we can assume  $e = 0$  due to a translation. Then we obtain the systems

$$\dot{x} = k + cx + dy, \quad \dot{y} = l + fy - x^2, \quad (7.21)$$

for which

$$\mu_{0,1} = 0, \quad \mu_2 = d^2x^2 = \tilde{G} \neq 0, \quad K = 0 = \tilde{N}, \quad L_2 = 0 = \tilde{S}_2. \quad (7.22)$$

**Case  $\mu_2 < 0$ .** From (7.20) and (7.22) it follows that the condition  $\mu_2 < 0$  implies  $\tilde{N} \neq 0$  and  $\tilde{S}_2 > 0$ . Hence we obtain the conditions for Figure 40 and we conclude that the condition  $\mu_2 < 0$  immediately leads to the conditions for Figure 40.

**Case  $\mu_2 > 0$  ( i.e.  $\tilde{G} > 0$ ).** Assume that the condition  $K \neq 0$  holds (then  $\tilde{N} \neq 0$ ). If  $K_2 < 0$  we have  $\tilde{S}_2 < 0$  and then we obtain the conditions for Figure 30. If either  $K_2 > 0$  or  $K_2 = 0$  via  $\tilde{G} > 0$  in both cases we get Figure 32.

Suppose  $K = 0$  (i.e.  $\tilde{N} = 0$ ). In this case we obtain the conditions  $\tilde{G} \neq 0$ ,  $\tilde{N} = 0$  which lead to Figure 30. Note that from (7.20) and (7.22) it follows that the condition  $K = 0$  implies  $\mu_2 > 0$ . ■

**Lemma 7.4.** *Let  $\tilde{\mathfrak{C}}_3$  be the conjunction of the following conditions:  $\tilde{M} = \tilde{\mu} = \tilde{H} = \tilde{G} = \tilde{N} = 0$  and  $\tilde{L}\tilde{F} \neq 0$ . Let  $\mathfrak{C}_3$  be the conjunction of the following conditions:  $M = \mu_0 = \mu_1 = \mu_2 = K = 0$  and  $C_2\mu_3 \neq 0$ . We have the following equivalences:*

$$\begin{aligned} \text{Figure 31 : } & \left[ \begin{array}{l} \tilde{\mathfrak{C}}_3, \tilde{F} \neq 0, (\tilde{S}_3 = 0) \\ \vee (\tilde{F}\tilde{S}_1 > 0, \tilde{S}_3 > 0) \end{array} \right] & \Leftrightarrow & \mathfrak{C}_3, \mu_3K_1 > 0, K_3 \geq 0; \\ \text{Figure 33 : } & \tilde{\mathfrak{C}}_3, \tilde{F}\tilde{S}_1 < 0, \tilde{S}_3 < 0 & \Leftrightarrow & \mathfrak{C}_3, \mu_3K_1 < 0; \\ \text{Figure 38 : } & \tilde{\mathfrak{C}}_3, \tilde{F}\tilde{S}_1 > 0, \tilde{S}_3 < 0 & \Leftrightarrow & \mathfrak{C}_3, \mu_3K_1 > 0, K_3 < 0. \end{aligned}$$



*Proof:* We are in the class of systems  $(\mathbf{S}_N)$  for which we must set the conditions  $\mu_0 = \mu_1 = \mu_2 = 0 = K$ ,  $\mu_3 \neq 0$ . We have  $\mu_0 = -h^3 = 0$  hence  $h = 0$  and then  $K = 2gx^2$ . The condition  $K = 0$  yields  $g = 0$  and this leads to the systems (7.21) for which the condition  $\mu_2 = d^2x^2 = 0$  yields  $d = 0$ . Hence we obtain the systems

$$\dot{x} = k + cx, \quad \dot{y} = l + fy - x^2, \quad (7.23)$$

for which

$$\begin{aligned} \mu_{0,1,2} &= 0, \quad \mu_3 = -c^2fd^2x^3 = \tilde{F} \neq 0, \quad K = 0 = \tilde{N}, \\ K_1 &= -cx^3 = \tilde{S}_1, \quad K_3 = 6f(2c - f)x^6 = \tilde{S}_3. \end{aligned} \quad (7.24)$$

We note that  $\mu_3K_1 = c^3fx^6 \neq 0$  and hence  $\text{sign}(\mu_3K_1) = \text{sign}(cf) = \text{sign}(\tilde{F}\tilde{S}_1)$ .

**Case**  $\mu_3K_1 < 0$ . From (7.24) we obtain  $\tilde{S}_3 < 0$  and hence we conclude that the condition  $\mu_3K_1 < 0$  leads to the conditions for Figure 33.

**Case**  $\mu_3K_1 > 0$ . For  $K_3 < 0$  (then  $\tilde{S}_3 < 0$ ) we obtain the conditions for Figure 38. If either  $K_3 > 0$  or  $K_3 = 0$  we observe that in both cases we get the conditions for Figure 31. From (7.24) it follows that the condition  $K_3 = 0$  implies  $\mu_3K_1 > 0$ . Therefore we conclude that the conditions  $\mu_3K_1 > 0$  and  $K_3 \geq 0$  lead to the conditions for Figure 31. ■

**Lemma 7.5.** *Let  $\tilde{\mathfrak{C}}_4$  be the conjunction of the following conditions:  $\tilde{M} = \tilde{\mu} = \tilde{H} = \tilde{G} = \tilde{F} = 0$  and  $\tilde{L}\tilde{V} \neq 0$ . Let  $\mathfrak{C}_4$  be the conjunction of the following conditions:  $M = \mu_0 = \mu_1 = \mu_2 = \mu_3 = 0$  and  $C_2\mu_4 \neq 0$ . We have the following equivalences:*

$$\begin{aligned} \text{Figure 30 : } & \left[ \begin{array}{l} \tilde{\mathfrak{C}}_4, \tilde{V} \neq 0, (\tilde{N} \neq 0, \tilde{S}_3 > 0) \\ \vee (\tilde{N} = \tilde{S}_1 = \tilde{S}_3 = 0) \vee \\ (\tilde{N} = 0, \tilde{S}_1 \neq 0, \tilde{V} > 0) \end{array} \right] & \Leftrightarrow \quad \mathfrak{C}_4, \mu_4 > 0, K_3 \geq 0; \\ \text{Figure 32 : } & \tilde{\mathfrak{C}}_4, \tilde{V} \neq 0, \tilde{N} = \tilde{S}_1 = 0, \tilde{S}_3 \neq 0 & \Leftrightarrow \quad \mathfrak{C}_4, \mu_4 > 0, K_3 < 0, K = 0; \\ \text{Figure 35 : } & \tilde{\mathfrak{C}}_4, \tilde{V} < 0, \tilde{N} = 0, \tilde{S}_1 \neq 0 & \Leftrightarrow \quad \mathfrak{C}_4, \mu_4 < 0; \\ \text{Figure 36 : } & \tilde{\mathfrak{C}}_4, \tilde{V} \neq 0, \tilde{N} \neq 0, \tilde{S}_3 < 0 & \Leftrightarrow \quad \mathfrak{C}_4, \mu_4 > 0, K_3 < 0, K \neq 0. \end{aligned}$$

*Proof:* We are in the class of systems  $(\mathbf{S}_N)$  for which we must set the conditions  $\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0$ ,  $\mu_4 \neq 0$ . We have  $\mu_0 = -h^3 = 0$  which implies  $h = 0$  and then  $\mu_1 = dg^3x$  and  $K = 2gx^2$ . We shall consider two subcases:  $K \neq 0$  and  $K = 0$ .

If  $K \neq 0$  then the condition  $\mu_1 = 0$  leads to the systems (7.19) for which  $\mu_2 = kx^2$ . Hence the condition  $\mu_2 = 0$  yields  $k = 0$  and we calculate:  $\mu_3 = -clx^3$  and  $\mu_4 = -l(c^2x - lx - c^2y)x^3$ . Hence the conditions  $\mu_3 = 0$  and  $\mu_4 \neq 0$  yield  $c = 0$ ,  $l \neq 0$  and we obtain the systems

$$\dot{x} = x^2, \quad \dot{y} = l - x^2 + xy, \quad (7.25)$$

for which

$$\mu_{0,1,2,3} = 0, \quad \mu_4 = l^2 x^4 = \tilde{V} \neq 0, \quad K = \frac{1}{2} x^2 = 4\tilde{N}, \quad K_3 = -6lx^6 = \tilde{S}_3 \neq 0. \quad (7.26)$$

Admit now that  $K = 0$ . This leads to the systems (7.23) for which the condition  $\mu_3 = -c^2 f d^2 x^3 = 0$  yields  $cf = 0$ . Then we get the systems

$$\dot{x} = k + cx, \quad \dot{y} = l + fy - x^2, \quad (7.27)$$

with  $cf = 0$  and

$$\begin{aligned} \mu_{0,1,2,3} &= 0, \quad \mu_4 = (k^2 - c^2 l)x^4 = \tilde{V} \neq 0, \quad K = 0 = \tilde{N}, \\ K_1 &= -cx^3 = \tilde{S}_1, \quad K_3 = -6f^2 x^6 = \tilde{S}_3, \quad K_1 K_3 = 0. \end{aligned} \quad (7.28)$$

**Case**  $\mu_4 < 0$  (i.e.  $\tilde{V} < 0$ ). From (7.26) and (7.28) we obtain that the condition  $\mu_4 < 0$  implies  $\tilde{N} = 0$  and  $\tilde{S}_1 \neq 0$ . Hence for  $\mu_4 < 0$  we obtain the conditions for Figure 35.

**Case**  $\mu_4 > 0$ . Then  $\tilde{N} > 0$  and we shall consider 3 subcases:  $K_3 < 0$ ,  $K_3 > 0$  and  $K_3 = 0$ .

**Subcase**  $K_3 < 0$ . If  $K \neq 0$  then  $\tilde{N} \neq 0$  and we have the conditions for Figure 36. Suppose  $K = 0$ , i.e.  $\tilde{N} = 0$ . Then by  $K_3 \neq 0$  from (7.28) we have  $\tilde{S}_1 = 0$ . Therefore we conclude that conditions  $K_3 < 0$  and  $K = 0$  lead to the Figure 32.

**Subcase**  $K_3 > 0$ . Then  $\tilde{S}_3 > 0$  and from (7.26) and (7.28) we conclude that  $K \neq 0$ , i.e.  $\tilde{N} \neq 0$ . Hence we obtain one series of the conditions for Figure 30.

**Subcase**  $K_3 = 0$ . Then  $\tilde{S}_3 = 0$  and according to (7.26) and (7.28) we have  $K = 0$ . This leads to the systems (7.27) for which the condition  $K_3 = 0$  yields  $f = 0$ . Then we have either  $K_1 \neq 0$  (i.e.  $\tilde{S}_1 \neq 0$ ) or  $K_1 = 0$  (i.e.  $\tilde{S}_1 = 0$ ). Since the conditions  $\tilde{V} > 0$  and  $\tilde{S}_3 = 0$  hold, both cases lead to the conditions for Figure 30.

Lemma 7.5 is proved and this completes the proof of the step II. ■

*Proof of step III.* We draw the attention to the fact that all the constructed polynomials which were used in Theorems 5.1 and 7.1 are *GL*-comitants. But in fact we are interested in the action of the affine group  $\text{Aff}(2, \mathbb{R})$  on these systems. We shall prove the following lemma.

**Lemma 7.6.** *The polynomials which are used in Theorems 5.1 or 7.1 have the properties indicated in the Table 6. In the last column are indicated the algebraic sets on which the GL-comitants on the left are CT-comitants. The Table 6 shows us that all conditions included in the statements of Theorems 5.1 or 7.1 are affinely invariant.*

**Table 6**

Case	$GL$ -comitants	Degree in		Weight	Algebraic subset $V(*)$
		$a$	$x$ and $y$		
1	$\eta(a), \mu_0(a), \kappa(a)$	4	0	2	$V(0)$
2	$C_2(a, x, y)$	1	3	-1	$V(0)$
3	$K(a, x, y)$	2	2	0	$V(0)$
4	$L(a, x, y)$	2	2	0	$V(0)$
5	$M(a, x, y)$	2	2	0	$V(0)$
6	$N(a, x, y)$	2	2	0	$V(0)$
7	$R(a, x, y)$	2	2	0	$V(0)$
8	$\kappa_1(a)$	3	0	1	$V(\eta, \kappa)$
9	$\kappa_2(a)$	2	0	0	$V(\eta, \kappa, L, K_1)$
10	$K_2(a, x, y)$	4	2	0	$V(\eta, \mu_0, \mu_1, \kappa, \kappa_1)$
11	$K_3(a, x, y)$	4	6	-2	$V(M, \mu_0, \mu_1, \mu_2)$
12	$K_1(a, x, y)$	2	3	-1	$V(K)$
13	$\mu_1(a, x, y)$	4	1	1	$V(\mu_0)$
14	$\mu_2(a, x, y)$	4	2	0	$V(\mu_0, \mu_1)$
15	$\mu_3(a, x, y)$	4	3	-1	$V(\mu_0, \mu_1, \mu_2)$
16	$\mu_4(a, x, y)$	4	4	-2	$V(\mu_0, \mu_1, \mu_2, \mu_3)$

*Proof:* I. Cases 1–7. The polynomials  $\eta(a)$ ,  $\kappa(a)$ ,  $\mu_0(a)$ ,  $K(a, x, y)$ ,  $L(a, x, y)$ ,  $M(a, x, y)$ ,  $N(a, x, y)$  and  $R(a, x, y)$  are  $T$ -comitants, because these  $GL$ -comitants were constructed only by using the coefficients of the polynomials  $p_2(x, y)$  and  $q_2(x, y)$ .

II. Cases 8–11. a) We consider the  $GL$ -invariant  $\kappa_1(a)$  which according to Table 4 was used only in the class of systems  $(\mathbf{S}_{III})$ . It was shown before (see page 35) that for  $\kappa = 0$  the systems  $(\mathbf{S}_{III})$  can be brought by an affine transformation to the systems (7.6) for which  $\kappa_1 = -32d$ . On the other hand for any system in the orbit under the translation group action of a system (7.6) corresponding to a point  $\mathbf{a} \in \mathbb{R}^{12}$  we obtain  $\kappa_1(\mathbf{a}) = -32d$ . Hence the value of  $\kappa_1$  does not depend of the vector defining the translations. Therefore we conclude that the polynomial  $\kappa_1$  is a  $CT$ -comitant modulo  $\langle \eta, \kappa \rangle$ .

b) We consider now the  $GL$ -invariant  $\kappa_2(a)$ . From Table 4 we observe that  $\kappa_2(a)$  is only applied to distinguish the Figures 8 and 17 when for the systems  $(\mathbf{S}_{III})$  the conditions

$\kappa = L = K_1 = 0$  hold. As it was shown before (see page 35) for  $\kappa = L = K_1 = 0$  the systems  $(\mathbf{S}_{III})$  can be brought by an affine transformation to the systems (7.8) for which  $\kappa_2 = -k$ . On the other hand for any system in the orbit under the translation group action of a system (7.8) corresponding to a point  $\mathbf{a} \in \mathbb{R}^{12}$  we obtain  $\kappa_2(\mathbf{a}) = -k$ . Hence we conclude that the polynomial  $\kappa_2$  is a  $CT$ -comitant modulo  $\langle \eta, \kappa, L, K_1 \rangle$ .

c) We examine now the  $GL$ -invariant  $K_2(a)$  which was used in cases  $(\mathbf{S}_{III})$  and  $(\mathbf{S}_{IV})$ . Assume firstly  $\eta = 0$  and  $M \neq 0$ , i.e. we are in the class of the systems  $(\mathbf{S}_{III})$ . We have shown before (see page 35) that for  $\kappa = \kappa_1 = 0$  the systems  $(\mathbf{S}_{III})$  can be brought by an affine transformation to the systems (7.7) for which  $K_2 = 48(g^2 - g + 2)(c^2 - 4gk)x^2$ . Suppose now that the conditions  $M = 0$  and  $C_2 \neq 0$  hold, i.e. we are in the class of the systems  $(\mathbf{S}_{IV})$ . It was shown before (see page 36) that for  $\mu_0 = \mu_1 = 0$  the systems  $(\mathbf{S}_{IV})$  can be brought by an affine transformation to the systems (7.10) for which  $K_2 = 24g^2(c^2 - 8gk)x^2$ .

On the other hand for any system in the orbit under the translation group action of a system (7.7) (respectively, of a system (7.10)) corresponding to a point  $\mathbf{a} \in \mathbb{R}^{12}$  (respectively,  $\mathbf{a}_1 \in \mathbb{R}^{12}$ ) we obtain  $K_2(\mathbf{a}, x, y) = 48(g^2 - g + 2)(c^2 - 4gk)x^2$  (respectively,  $K_2(\mathbf{a}_1, x, y) = 24g^2(c^2 - 8gk)x^2$ ). Calculations yield that for the system (7.7) (respectively, for the system (7.10)) we have  $\mu_0 = \mu_1 = 0$  (respectively  $\kappa = \kappa_1 = 0$ ). Hence we conclude that the  $GL$ -comitant  $K_2(a, x, y)$  is a  $CT$ -comitant modulo  $\langle \eta, \mu_0, \mu_1, \kappa, \kappa_1 \rangle$ .

d). We examine now the comitant  $K_3$  which is applied for systems  $(\mathbf{S}_{IV})$  only in the cases when  $\Delta_S \geq 3$ , i.e.  $\mu_0 = \mu_1 = \mu_2 = 0$ . It was shown before (see page 36) that for  $\mu_0 = \mu_1 = \mu_2 = 0$  the systems  $(\mathbf{S}_{IV})$  can be brought by an affine transformation either to the systems (7.11) for  $K \neq 0$  or to the systems (7.12) for  $K = 0$ . Calculations yield, that for any system in the orbit under the translation group action of a system (7.11) (respectively, of a system (7.12)) corresponding to a point  $\mathbf{a} \in \mathbb{R}^{12}$  (respectively,  $\mathbf{a}_1 \in \mathbb{R}^{12}$ ) we obtain  $K_3(\mathbf{a}, x, y) = -12g^2lx^6$  (respectively,  $K_3(\mathbf{a}_1, x, y) = 3f(2c - f)x^6$ ). Hence in both cases the values of  $K_3$  do not depend of the vector defining the translations. Therefore the  $GL$ -comitant  $K_3(a, x, y)$  is a  $CT$ -comitant modulo  $\langle M, \mu_0, \mu_1, \mu_2 \rangle$ .

III) *The cases 12–16.* Let  $\tau \in T(2, \mathbb{R})$  be the translation:  $x = \tilde{x} + \alpha$ ,  $y = \tilde{y} + \beta$  and consider a quadratic system (3.1) which corresponds to a point  $\mathbf{a} \in \mathbb{R}^{12}$ . It is sufficient to verify that the following relations occur, where  $\xi = \tilde{x}\beta - \tilde{y}\alpha$ :

$$K_1(r_\tau \cdot \mathbf{a}, \tilde{x}, \tilde{y}) = K_1(\mathbf{a}, \tilde{x}, \tilde{y}) - \xi K(\mathbf{a}, \tilde{x}, \tilde{y});$$

$$\mu_s(r_\tau \cdot \mathbf{a}, \tilde{x}, \tilde{y}) = \mu_s(\mathbf{a}, \tilde{x}, \tilde{y}) + \sum_{k=0}^{s-1} \binom{4-k}{s-k} \xi^{s-k} \mu_k(\mathbf{a}, \tilde{x}, \tilde{y}), \quad s = 1, 2, 3, 4.$$

So, Lemma 7.6 is proved and this completes the proof of the Theorem 7.1. ■

## References

- [1] J. C. Artés and J. Llibre, *Quadratics Hamiltonian Vector Fields*. Journal of Diff. Eq. **107** (1994), 80-95.
- [2] J. C. Artés and J. Llibre, *Quadratic vector fields with a weak focus of third order*. Publicacions Matemàtiques, **41** (1997), 7-39.
- [3] N.N. Bautin, E.A. Leontovich, *Methods and aspects of the qualitative study of dynamical systems on the plane*, "Nauka", Moscow, 1976 (Russian).
- [4] V. A. Baltag, N. I. Vulpe, *Affine-invariant conditions for determining the number and multiplicity of singular points of quadratic differential systems*. Izv. Akad. Nauk Respub. Moldova Mat. 1993, no. 1, 39-48
- [5] V. A. Baltag, N. I. Vulpe, *Total multiplicity of all finite critical points of the polynomial differential system*. Planar nonlinear dynamical systems (Delft, 1995). Differential Equations & Dynam. Systems **5** (1997), no. 3-4, 455-471.
- [6] D. Boullaras, Iu. Calin. L. Timochouk, N. Vulpe, *T-comitants of quadratic systems: A study via the translation invariants*. Delft University of Technology, Faculty of Technical Mathematics and Informatics, Report no. 96-90, 1996; (URL: <ftp://ftp.its.tudelft.nl/publications/tech-reports/1996/DUT-TWI-96-90.ps.gz>)
- [7] Iu. T. Calin, N. I. Vulpe, *Affine-invariant conditions for the topological discrimination of quadratic Hamiltonian differential systems*. (Russian) Differ. Uravn. **34** (1998), no. 3, 298-302; translation in Differential Equations **34** (1998), no. 3, 297-301.
- [8] C. Camacho, *Complex foliations arising from Polynomial Differential Equations*. Notes by Maria Izabel Camacho, in Bifurcations and Periodic Orbits of Vector Fields, (Edited by D. Schlomiuk), 1-19, Kluwer Academic Publishers, (1993).
- [9] Armengol Gasull, Sheng Li-Ren, Jaume Llibre, *Chordal quadratic systems*. Rocky Mountain Journal of Mathematics, **16** (1986), Number 4, 751-781.
- [10] E. A. Gonzales Velasco, *Generic properties of polynomial vector fields at infinity*. Trans. A.M.S., **143** (1969), 201-222.
- [11] W. Fulton, *Algebraic curves. An introduction to Algebraic Geometry*, W.A. Benjamin, Inc., New York, 1969.
- [12] J. H. Grace and A. Young. *The algebra of invariants*. New York: Stechert, 1941.
- [13] R. Kooij, R. E. Reyn, *Infinite singular points of quadratic systems in the plane*. Nonlinear Analysis, Theory and Applications, **24** (1995), no. 6, 895-927.

- [14] J. Llibre, D. Schlomiuk, *The geometry of quadratic systems with a weak focus of third order*. To appear in the Canadian J. of Math. (A previous version of this paper appeared as Preprint, núm. 486, Nov. 2001. CRM, Barcelona, 48 pp.)
- [15] L. Markus, *Global structure of ordinary differential equations in the plane*, Trans. Amer. Mat. Soc. **76** (1954), 127-148.
- [16] I. Nikolaev, N. Vulpe, *Topological classification of quadratic systems at infinity*. Journal of the London Mathematical Society, **2** (1997), no. 55, 473-488.
- [17] P. J. Olver, *Classical Invariant Theory*, (London Mathematical Society student texts: 44), Cambridge University Press, 1999.
- [18] J. Pal, D. Schlomiuk, *Summing up of the dynamics of Quadratic Hamiltonian Systems with a center*. Canadian Journal of Mathematics, **49** (1997), no. 3, 583-599.
- [19] J. Pal and D. Schlomiuk, *Multiplicity of intersection and limit cycles in quadratic systems with a weak focus*. Preprint, September 1999.
- [20] H. Poincaré, *Mémoire sur les courbes définies par les équations différentielles*, J. Math. Pures Appl. (4) **1** (1885), 167-244; Œuvres de Henri Poincaré, Vol. **1**, Gauthier-Villard, Paris, 1951, pp 95-114.
- [21] M.N. Popa, *Applications of algebras to differential systems*, Academy of Science of Moldova, 2001 (Russian), 224 pp.
- [22] M. N. Popa, K. S. Sibirsky, *Affine classification of a system with quadratic nonlinearities and not single valued canonical form*. (Russian) Differ. Uravn. **14** (1978), no. 6, 1028-1033.
- [23] D. Schlomiuk, *Algebraic particular integrals, integrability and the problem of the center*. Transactions of the A.M.S., **338** (1993), Number 2, August, 799-841.
- [24] D. Schlomiuk, *Basic algebro-geometric concepts in the study of planar polynomial vector fields*, Publications Mathématiques, Vol **41** (1997), 269-295.
- [25] D. Schlomiuk, *Aspects of planar polynomial vector fields: global versus local, real versus complex, analytic versus algebraic and geometric*, to appear in "Normal forms, bifurcations and finiteness problems in nonlinear dynamical systems", Yu. Il'ashenko and C. Rousseau editors., 40 pp.
- [26] D. Schlomiuk, J. Pal, *On the Geometry in the Neighborhood of Infinity of Quadratic Differential Systems with a Weak Focus*. Qualitative Theory of Dynamical Systems, **2** (2001), no. 1, 1-43
- [27] D. Schlomiuk, N. Vulpe, *Geometry of quadratic differential systems in the neighbourhood of the line at infinity*, CRM Report no. 2701, Université de Montréal, 2001, 41 pp.
- [28] K. S. Sibirsky, *The method of invariants in the qualitative theory of differential equations*. Kishinev: RIO AN Moldavian SSR, 1968.
- [29] K.S. Sibirsky, *Algebraic invariants of differential equations and matrices*. Kishinev: Shtiintsa, 1976 (in Russian). 268 pp.
- [30] K. S. Sibirsky, *Introduction to the algebraic theory of invariants of differential equations*. Translated from the Russian. Nonlinear Science: Theory and Applications. Manchester University Press, Manchester, 1988, 169 pp.

- [31] N.I. Vulpe, *Affine-invariant conditions for the topological discrimination of quadratic systems with a center*. Translated from *Differentsial'nye Uravneniya*, **19** (1983), No.3, 371-379.
- [32] N.I.Vulpe, *Polynomial bases of comitants of differential systems and their applications in qualitative theory*. (Russian) "Shtiintsa", Kishinev, 1986, 172 p.

## APPENDIX

Let us consider the tensorial form of quadratic system:

$$\frac{dx^j}{dt} = a^j + a_\alpha^j x^\alpha + a_{\alpha\beta}^j x^\alpha x^\beta \quad (j, \alpha, \beta = 1, 2).$$

The following invariants and comitants, defined by polynomials of  $J_i, R_i$  which are tensorially defined  $GL$ -comitants, were used in [16] for the classification in the neighbourhood of infinity of quadratic differential systems:

$$\begin{aligned} 2\tilde{\mu} &= J_4, \quad \tilde{\sigma} = J_7, \quad 2\tilde{\theta} = J_5, \quad \tilde{L} = R_{12}, \quad 2\tilde{M} = 9R_3 + 6R_6 - 8R_{11}^2, \quad \tilde{S}_1 = R_5, \\ \tilde{S}_2 &= 2J_1^2 R_6 + 2J_1 R_1^2 - 2J_2 R_6 + J_2 R_{11}^2 + 8J_3 R_3 - 8J_3 R_6 - 4R_7 - R_8, \quad \tilde{H} = R_{13}, \\ \tilde{S}_3 &= R_{12}^2 (7J_2 - 6J_1^2 - 8J_3) - R_{12} (10J_1 R_5 + 4R_1 R_{10} - 6R_3 R_9) + 4R_3 R_{10}^2 - 4R_5^2, \\ \tilde{S}_4 &= 4J_3 - J_2, \quad \tilde{V} = R_4^2 - R_2 R_5, \quad 2\tilde{A} = 2R_6 - 3R_3, \quad 2\tilde{\eta} = J_4 + 20J_5 - 8J_6 \\ 2\tilde{N} &= R_3, \quad 2\tilde{G} = 2R_1^2 - 2J_2 R_3 + 4R_7 + R_8, \quad 2\tilde{F} = J_2 R_5 + 4R_2 R_3 + 4R_1 R_4, \end{aligned}$$

where

$$\begin{aligned} J_1 &= a_\alpha^\alpha, \quad J_2 = a_p^\alpha a_q^\beta \varepsilon_{\alpha\beta} \varepsilon^{pq}, \quad J_3 = a^\alpha a_{\alpha\beta}^\beta, \quad J_4 = a_{pr}^\alpha a_{qk}^\beta a_{sn}^\gamma a_{lm}^\delta \varepsilon_{\alpha\beta} \varepsilon_{\gamma\delta} \varepsilon^{pq} \varepsilon^{rs} \varepsilon^{kl} \varepsilon^{mn}, \\ J_5 &= a_\gamma^\alpha a_{\delta r}^\beta a_{qk}^\gamma a_{sl}^\delta \varepsilon_{\alpha\beta} \varepsilon^{pq} \varepsilon^{rs} \varepsilon^{kl}, \quad J_6 = a_{pr}^\alpha a_{\alpha q}^\beta a_{\delta s}^\gamma a_{\beta\gamma}^\delta \varepsilon^{pq} \varepsilon^{rs}, \quad J_7 = a_p^\alpha a_{\gamma q}^\beta a_{\alpha\beta}^\gamma \varepsilon^{pq}, \\ R_1 &= x^\alpha a_q^\beta a_{p\alpha}^\gamma \varepsilon_{\beta\gamma} \varepsilon^{pq}, \quad R_2 = x^\alpha a^\beta a_\alpha^\gamma \varepsilon_{\beta\gamma}, \quad R_3 = x^\alpha x^\beta a_{p\alpha}^\gamma a_{q\beta}^\delta \varepsilon_{\gamma\delta} \varepsilon^{pq}, \quad R_4 = x^\alpha x^\beta a^\gamma a_{\alpha\beta}^\delta \varepsilon_{\gamma\delta}, \\ R_5 &= x^\alpha x^\beta x^\gamma a_\alpha^\delta a_{\beta\gamma}^\mu \varepsilon_{\delta\mu}, \quad R_6 = x^\alpha x^\beta a_{\alpha\beta}^\gamma a_{\gamma\delta}^\delta, \quad R_7 = x^\alpha x^\beta a^\gamma a_{\alpha p}^\delta a_{\beta s}^\mu a_{qr}^\nu \varepsilon_{\gamma\delta} \varepsilon_{\mu\nu} \varepsilon^{pq} \varepsilon^{rs}, \\ R_8 &= x^\alpha x^\beta a_\alpha^\gamma a_{\beta p}^\delta a_{qr}^\mu a_{qs}^\nu \varepsilon_{\gamma\mu} \varepsilon_{\delta\nu} \varepsilon^{pq} \varepsilon^{rs}, \quad R_9 = x^\alpha a^\beta \varepsilon_{\beta\alpha}, \quad R_{10} = x^\alpha x^\beta a_\alpha^\gamma \varepsilon_{\gamma\beta}, \\ R_{11} &= x^\alpha a_{\alpha\beta}^\beta, \quad R_{12} = x^\alpha x^\beta x^\gamma a_{\alpha\beta}^\delta \varepsilon_{\delta\gamma}, \quad R_{13} = x^\alpha a_p^\beta a_{\alpha r}^\gamma a_{qk}^\delta a_{sl}^\mu \varepsilon_{\beta\gamma} \varepsilon_{\delta\mu} \varepsilon^{pq} \varepsilon^{rs} \varepsilon^{kl}, \end{aligned}$$

and

$$\varepsilon^{11} = \varepsilon^{22} = \varepsilon_{11} = \varepsilon_{22} = 0, \quad \varepsilon^{12} = \varepsilon_{12} = -\varepsilon^{21} = -\varepsilon_{21} = 1.$$